LEGENDRE POLYNOMIALS AND APPLICATIONS

1. Exercises.

Exercise 1. Use the recurrence relation that gives the coefficients of the Legendre polynomials to show that

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} .$$

We have $P_{2n}(x) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \frac{(-1)^k (4n-2k)!}{k!\,(2n-k)!\,(2n-2k)!} x^{2n-2k}$. Therefore

$$P_{2n}(0) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \frac{(-1)^k (4n-2k)!}{k!\,(2n-k)!\,(2n-2k)!} 0^{2n-2k} = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} .$$

Exercise 2. Use exercise 1 to verify that

$$P_{2n}(0) - P_{2n+2}(0) = (-1)^n \left(\frac{4n+3}{2n+2}\right) \frac{(2n)!}{2^{2n}(n!)^2} .$$

Exercise 3. Use $P_0(x) = 1$, $P_1(x) = x$ and the recurrence relation (8):

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

to find $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

We have

$$P_2(x) = \frac{3}{2}xP_1(x) - \frac{1}{2}xP_0(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_3(x) = \frac{5}{3}xP_2(x) - \frac{2}{3}xP_1(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

$$P_4(x) = \frac{7}{4}xP_3(x) - \frac{3}{4}xP_2(x) = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$$

Exercise 4. Use Rodrigues' formula to find the Legendre polynomials $P_0(x)$ to $P_5(x)$.

Exercise 5. Use The relations

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$
$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$

to establish the formula

$$xP'_n(x) = nP_n(x) + P'_{n-1}(x) .$$

Exercise 6. Write x^2 as a linear combination of $P_0(x)$, $P_1(x)$, and $P_2(x)$. That is, find constants A, B, and C so that

$$x^2 = AP_0(x) + BP_1(x) + CP_2(x) .$$

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By using $P_0(x) = 1,\ P_1(x) = x$ and $P_2(x) = \dfrac{3}{2}x^2 - \dfrac{1}{2}$, we get

$$\begin{aligned} x^2 &= AP_0(x) + BP_1(x) + CP_2(x) \\ &= A + Bx + C\left(\frac{3}{2}x^2 - \frac{1}{2}\right) \\ &= \left(A - \frac{C}{2}\right) + Bx + \frac{3C}{2}x^2 \end{aligned}$$

After equating the coefficients, we get $C = \dfrac{2}{3}$, $B = 0$, $A = \frac{1}{3}$. Hence

$$x^2 = \frac{1}{3}P_0(x) + 0P_1(x) + \frac{2}{3}P_2(x)\ .$$

Exercise 7. Write x^3 as a linear combination of P_0, P_1, P_2, and P_3.

Exercise 8. Write x^4 as a linear combination of P_0, P_1, P_2, P_3 and P_4.

An argument similar to that used in Problem 6 shows that

$$x^4 = \frac{1}{5}P_0(x) + 0P_1(x) + \frac{4}{7}P_2(x) + 0P_3(x) + \frac{8}{35}P_4(x)$$

Exercise 9. Use the results from exercises 6, 7, and 8 to find the integrals.

$$\begin{aligned} &\int_{-1}^{1} x^2P_2(x)dx, \qquad \int_{-1}^{1} x^2P_{31}(x)dx \\ &\int_{-1}^{1} x^3P_1(x)dx, \qquad \int_{-1}^{1} x^3P_4(x)dx \\ &\int_{-1}^{1} x^4P_2(x)dx, \qquad \int_{-1}^{1} x^4P_4(x)dx \end{aligned}$$

We use the fact that orthogonality $\langle P_n(x), P_m(x)\rangle = 0$ (for $n \neq m$) and norm $\langle P_n(x), P_n(x)\rangle = 2/(2n+1)$ and the above exercises to find

$$\begin{aligned} \int_{-1}^{1} x^2P_2(x) &= \langle\frac{1}{3}P_0(x) + 0P_1(x) + \frac{2}{3}P_2(x), P_2(x)\rangle \\ &= \frac{1}{3}\langle P_0(x), P_2(x)\rangle + \frac{2}{3}\|P_2(x)\|^2 = \frac{4}{15} \\ \int_{-1}^{1} x^2P_{31}(x) &= \langle\frac{1}{3}P_0(x) + 0P_1(x) + \frac{2}{3}P_2(x), P_{31}(x)\rangle \\ &= \frac{1}{3}\langle P_0(x), P_{31}(x)\rangle + \frac{2}{3}\langle P_2(x), P_{31}(x)\rangle = 0 \end{aligned}$$

Exercise 10. Use the fact that for $m \in \mathbb{Z}^+$, the function x^m can written as a linear combination of $P_0(x), \cdots, P_m(x)$ to show that

$$\int_{-1}^{1} x^mP_n(x) = 0, \qquad \text{for } n > m\ .$$

Exercise 11. Use formula (6) and a property (even/odd) of the Legendre polynomials to verify that

$$\int_{-1}^{h} P_n(x)dx = \frac{1}{2n+1}[P_{n+1}(h) - P_{n-1}(h)]$$
$$\int_{h}^{1} P_n(x)dx = \frac{1}{2n+1}[P_{n-1}(h) - P_{n+1}(h)]$$

Note that since $P_m(1) = 1$ for all m and since P_m is an even/odd function if m is even/odd, then $P_m(-1) = (-1)^m$. It follows that $P_{n+1}(-1) - P_{n-1}(-1) = 0$. We have then

$$\begin{aligned}\int_{-1}^{h} P_n(x)dx &= \frac{1}{2n+1}\int_{-1}^{h}\left[P'_{n+1}(x) - P'_{n-1}(x)\right] dx \\ &= \frac{1}{2n+1}[P_{n+1}(h) - P_{n-1}(h) - P_{n+1}(-1) + P_{n-1}(-1)] \\ &= \frac{1}{2n+1}[P_{n-1}(h) - P_{n+1}(h)]\end{aligned}$$

Exercise 12. Find the Legendre series of the functions

$$f(x) = -3, \quad g(x) = x^3, \quad h(x) = x^4, \quad m(x) = |x|.$$

Legendre series of $|x|$: Since the function is even, then for $x \in [-1,\ 1]$ we have

$$|x| = \sum_{j=0}^{\infty} c_{2j}P_2j(x), \quad \text{with} \quad c_{2j} = (4j+1)\int_0^1 xP_{2j}(x)dx\,.$$

It follows from property (8): $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$ and Exercise 11 with $h = 0$ that

$$\begin{aligned}(4j+1)\int_0^1 xP_{2j}(x)dx &= \int_0^1 \left[(2j+1)P_{2j+1} + 2jP_{2j-1}(x)\right] dx \\ &= \frac{2j+1}{2j+2}\,[P_{2j}(0) - P_{2j+2}(0)] + \frac{2j}{2j}\,[P_{2j-2}(0) - P_{2j}(0)] \\ &= P_{2j-2}(0) - \frac{1}{2j+2}P_{2j}(0) - \frac{2j+1}{2j+2}P_{2j+2}(0)\end{aligned}$$

Therefore

$$|x| = \sum_{j=0}^{\infty}\left[P_{2j-2}(0) - \frac{1}{2j+2}P_{2j}(0) - \frac{2j+1}{2j+2}P_{2j+2}(0)\right] P_{2j}(x)$$

Exercise 13. Find the Legendre series of the function

$$f(x) = \begin{cases} 0 & \text{for } \ -1 < x < 0 \\ x & \text{for } \ 0 < x < 1 \end{cases}$$

Exercise 14. Find the Legendre series of the function

$$f(x) = \begin{cases} 0 & \text{for } \ -1 < x < h \\ 1 & \text{for } \ h < x < 1 \end{cases}$$

(Use exercise 11.)

Exercise 15. Find the first three nonzero terms of the Legendre series of the functions $f(x) = \sin x$ and $g(x) = \cos x$.

Legendre series of $\sin x$**:**

$$\sin x = \sum_{k=0}^{\infty} c_{2k+1} P_{2k+1}(x), \quad \text{with} \quad c_{2k+1} = (4k+3)\int_0^1 \sin x \, P_{2k+1}(x) dx \, .$$

We have

$$\begin{aligned} c_1 &= 3\int_0^1 x \sin x \, dx \\ c_3 &= \frac{7}{2}\int_0^1 (5x^3 \sin x - 3\sin x)\, dx \\ c_5 &= \frac{11}{8}\int_0^1 (63x^5 \sin x - 70x^3 \sin x + 15x \sin x)\, dx \, . \end{aligned}$$

Now we use the approximation $\sin 1 = 0.8415$ and $\cos 1 = 0.5403$ to find

$$\int_0^1 x \sin x \, dx \approx 0.301, \quad \int_0^1 x^3 \sin x \, dx \approx 0.177, \quad \int_0^1 x^5 \sin x \, dx \approx -2.036 \, .$$

It follows that

$$c_1 \approx 0.904, \quad c_3 \approx -0.0630 \quad c_5 \approx -187.214$$

and

$$\sin x \approx (0.904)P_1(x) - (0.0630)P_3(x) - (187.214)P_5(x) + \cdots$$

In exercises 16 to 19 solve the following Dirichlet problem inside the sphere

$$\begin{cases} u_{\rho\rho} + \dfrac{2}{\rho}u_\rho + \dfrac{1}{\rho^2}u_{\phi\phi} + \dfrac{\cos\phi}{\rho^2 \sin\phi}u_\phi = 0, & 0 < \rho < L, \quad 0 < \phi < \pi \\ u(L,\phi) = f(\phi) & 0 < \phi < \pi \end{cases}$$

Assume $u(\rho,\phi)$ is bounded.

For such a problem the bounded solutions with separated variables have the form $u_n(\rho,\phi) = \rho^n P_n(\cos\phi)$ with $n = 0,\ 1,\ 2,\ 2,\ \cdots$. The series representation of the general solution is

$$u(\rho,\phi) = \sum_{n=0}^{\infty} c_n \rho^n P_n(\cos\phi)\, .$$

The nonhomogeneous condition implies that $f(\phi) = \sum_{n=0}^{\infty} c_n L^n P_n(\cos\phi)$. Equivalently $g(x) = \sum_{n=0}^{\infty} c_n L^n P_n(x)$, where $g(x) = f(\arccos x)$. This means

$$L^n c_n = \frac{2n+1}{2}\int_{-1}^1 f(\arccos x) P_n(x)\, dx.$$

Exercise 16. $L = 10$, and $f(\phi) = \begin{cases} 50 & \text{for } \ 0 < \phi < (\pi/2), \\ 100 & \text{for } \ (\pi/2) < \phi < \pi. \end{cases}$ In this case we have (after using Exercise 11)

$$\begin{aligned} 10^n c_n &= \frac{2n+1}{2}\left[100\int_{-1}^0 P_n(x)\, dx + 50\int_0^1 P_n(x)\, dx\right] \\ &= 25\left[P_{n+1}(0) - P_{n-1}(0)\right] \end{aligned}$$

For n even we have $P_{n+1}(0) = P_{n-1}(0) =$ and $c_{\text{even}} = 0$. For $n = 2k+1$, we have (after using the formula $P_{2j} = \dfrac{(-1)^j(2j)!}{2^{2j}(j!)^2}$)

$$10^{2k+1}c_{2k+1} = \frac{(-1)^k(2k)!}{2^{2k+1}(k!)^2(k+1)}$$

The solution of the BVP is

$$u(\rho,\phi) = \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!}{2^{2k+1}(k!)^2(k+1)} \left(\frac{\rho}{10}\right)^{2k+1} P_{2k+1}(\cos\phi)\,.$$

Exercise 17. $L = 1$ and $f(\phi) = \cos\phi$.

Exercise 18. $L = 5$ and $f(\phi) = \begin{cases} 50 & \text{for } \; 0 < \phi < (\pi/4), \\ 0 & \text{for } \; (\pi/4) < \phi < \pi. \end{cases}$

Exercise 19. $L = 2$ and $f(\phi) = \sin^2\phi = 1 - \cos^2\phi$. In this case the function $g(x)$ is given by $g(x) = 1 - x^2$. Its Legendre series (use previous exercises)

$$g(x) = P_0(x) - (\frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)) = \frac{2}{3}P_0(x) - \frac{2}{3}P_2(x).$$

That is $c_n = 0$ for $n \neq 0,\, 2$; $2^0c_0 = 2/3$ and $2^2c_2 = -2/3$. The solution of the BVP is

$$u(\rho,\phi) = \frac{2}{3}P_0(\phi) - \frac{2}{3}\frac{\rho^2}{2^2}P_2(\phi) = \frac{2}{3} + \frac{\rho^2}{12} - \frac{\rho^2}{4}\cos^2\phi\,.$$

Exercise 20. Solve the following Dirichlet problem in a hemisphere

$$\begin{cases} u_{\rho\rho} + \dfrac{2}{\rho}u_\rho + \dfrac{1}{\rho^2}u_{\phi\phi} + \dfrac{\cos\phi}{\rho^2\sin\phi}u_\phi = 0, & 0 < \rho < 1, \quad 0 < \phi < (\pi/2) \\ u(1,\phi) = 100 & 0 < \phi < (\pi/2) \\ u(\rho,\pi/2) = 0 & 0 < \rho < 1\,. \end{cases}$$

In this problem the domain is given by $0 < \rho < 1$, $0 < \phi < (\pi/2)$ and the bounded solutions with separated variables have the form $u_n(\rho,\phi) = \rho^n P_n(\cos\phi)$ with $n = 0,\, 1,\, 2,\, 2, \cdots$ and satisfy $u_n(\rho,\pi/2) = 0$. This implies that n must be an odd integer. The series representation of the general solution is

$$u(\rho,\phi) = \sum_{k=0}^{\infty} c_{2k+1}\rho^{2k+1}P_{2k+1}(\cos\phi)\,.$$

The nonhomogeneous condition implies that $100 = \sum_{k=0}^{\infty} c_{2k+1}P_{2k+1}(\cos\phi)$ for $0 < \phi < (\pi/2)$. Equivalently

$$100 = \sum_{k=0}^{\infty} c_{2k+1}P_{2k+1}(t) \quad \forall t \in [0,\; 1]$$

(odd Legendre series of 100). Therefore

$$c_{2k+1} = (4k+3)\int_0^1 100P_{2k+1}(t)dt = 100[P_{2k}(0) - P_{2k+2}(0)] = 100\frac{(-1)^k(4k+3)\,(2k)!}{2^{k+1}(k+1)\,(k!)^2}$$

The solution of the BVP is

$$u(\rho,\phi) = 100\sum_{k=0}^{\infty} \frac{(-1)^k(4k+3)\,(2k)!}{2^{k+1}(k+1)\,(k!)^2}\,\rho^{2k+1}\,P_{2k+1}(\cos\phi)$$

Exercise 21. Solve the following Dirichlet problem in a hemisphere

$$\begin{cases} u_{\rho\rho} + \dfrac{2}{\rho} u_\rho + \dfrac{1}{\rho^2} u_{\phi\phi} + \dfrac{\cos\phi}{\rho^2 \sin\phi} u_\phi = 0, & 0 < \rho < 1, \quad 0 < \phi < (\pi/2) \\ u(1,\phi) = \cos\phi & 0 < \phi < (\pi/2) \\ u(\rho, \pi/2) = 0 & 0 < \rho < 1\,. \end{cases}$$

Exercise 22. Solve the following Dirichlet problem in a spherical shell

$$\begin{cases} u_{\rho\rho} + \dfrac{2}{\rho} u_\rho + \dfrac{1}{\rho^2} u_{\phi\phi} + \dfrac{\cos\phi}{\rho^2 \sin\phi} u_\phi = 0, & 1 < \rho < 2, \quad 0 < \phi < \pi \\ u(1,\phi) = 50 & 0 < \phi < \pi \\ u(2,\phi) = 100 & 0 < \phi < \pi\,. \end{cases}$$

Exercise 23. Solve the following Dirichlet problem in a spherical shell

$$\begin{cases} u_{\rho\rho} + \dfrac{2}{\rho} u_\rho + \dfrac{1}{\rho^2} u_{\phi\phi} + \dfrac{\cos\phi}{\rho^2 \sin\phi} u_\phi = 0, & 1 < \rho < 2, \quad 0 < \phi < \pi \\ u(1,\phi) = \cos\phi & 0 < \phi < \pi \\ u(2,\phi) = \sin^2\phi & 0 < \phi < \pi\,. \end{cases}$$

In this problem the domain is given by $1 < \rho < 1$, $0 < \phi < \pi$ and the bounded solutions with separated variables have the form $u_n(\rho,\phi) = \rho^n P_n(\cos\phi)$ and $u_n(\rho,\phi) = \rho^{-(n+1)} P_n(\cos\phi)$ with $n = 0, 1, 2, 2, \cdots$. The series representation of the general solution is

$$u(\rho,\phi) = \sum_{n=0}^{\infty} \left(a_n \rho^n + b_n \rho^{-(n+1)}\right) P_n(\cos\phi)\,.$$

The nonhomogeneous condition gives

$$\begin{aligned} u(1,\phi) &= \cos\phi = \sum_{n=0}^{\infty} (a_n + b_n)\, P_n(\cos\phi) \\ u(2,\phi) &= \sin^2\phi = \sum_{n=0}^{\infty} \left(2^n a_n + \frac{b_n}{2^{n+1}}\right) P_n(\cos\phi) \end{aligned}$$

Equivalently

$$\begin{aligned} t &= \sum_{n=0}^{\infty} (a_n + b_n)\, P_n(t) \\ 1 - t^2 &= \sum_{n=0}^{\infty} \left(2^n a_n + \frac{b_n}{2^{n+1}}\right) P_n(t) \end{aligned}$$

Since the Legendre series of t and $1 - t^2$ are

$$t = P_1(t), \quad \text{and} \quad 1 - t^2 = \frac{2}{3} P_0(t) - \frac{2}{3} P_2(t)\,,$$

it follows that the coefficients a_n's and b_n's satisfy

$$a_n + b_n = 0, \quad 2^n a_n + \frac{b_n}{2^{n+1}} = 0 \quad \text{for} \quad n \neq 0,\ 1,\ 2\,;$$

$$a_0 + b_0 = 0, \quad a_0 + \frac{b_0}{2} = \frac{2}{3}\,;$$

$$a_1 + b_1 = 1, \quad 2a_1 + \frac{b_1}{4} = 0\,;$$

$$a_2 + b_2 = 0, \quad 4a_2 + \frac{b_2}{8} = -\frac{2}{3}\,;$$

Solving these systems gives

$$a_0 = -b_0 = \frac{4}{3}, \quad a_1 = -\frac{1}{7}, \quad b_1 = \frac{8}{7}, \quad a_2 = -b_2 = \frac{-16}{99},$$

and $a_n = b_n = 0$ for $n \geq 3$. The solution of the BVP is

$$u(\rho, \phi) = \frac{4}{3}\left(1 - \frac{1}{\rho}\right) - \frac{1}{7}\left(\rho - \frac{8}{\rho^2}\right) P_1(\cos\phi) - \frac{16}{99}\left(\rho^2 - \frac{1}{\rho^3}\right) P_2(\cos\phi)\,.$$

Exercise 24. Find the gravitational potential at any point outside the surface of the earth knowing that the radius of the earth is 6400 km and that the gravitational potential on the Earth surface is given by

$$f(\phi) = \begin{cases} 200 - \cos\phi & \text{for} \quad 0 < \phi < (\pi/2), \\ 200 & \text{for} \quad (\pi/2) < \phi < \pi. \end{cases}$$

(This is an exterior Dirichlet problem)

The gravitational potential $V(\rho, \phi)$ satisfies the BVP

$$\begin{array}{ll} \triangle V(\rho, \phi) = 0 & \text{for} \quad \rho > R, \quad 0 < \phi < \pi \\ V(R, \phi) = f(\phi) & \lim\limits_{\rho\to\infty} V(\rho, \phi) = 0 \end{array}$$

where $R = 6400$ is the Earth radius. We know that the solutions of $\triangle V(\rho, \phi) = 0$ are $\rho^n P_n(\cos\phi)$ and $\rho^{-(n+1)} P_n(\cos\phi)$. Since $\lim\limits_{\rho\to\infty} \rho^n = \infty$, we seek a sulution in the form

$$V(\rho, \phi) = \sum_{n=0}^{\infty} c_n \rho^{-(n+1)} P_n(\cos\phi)\,.$$

The nonhomogeneous condition implies

$$f(\phi) = \sum_{n=0}^{\infty} \frac{c_n}{R^{(n+1)}} P_n(\cos\phi)$$

and so

$$\frac{c_n}{R^{(n+1)}} = \frac{2n+1}{2} \int_{-1}^{1} f(\arccos t) P_n(t)dt = \frac{2n+1}{2} \left[\int_{-1}^{1} 200 P_n(t)dt - \int_0^1 t P_n(t)dt\right]$$

For $n = 0$ we get

$$\frac{c_0}{R} = \frac{1}{2}\left[400 - \int_0^1 t dt\right] = \frac{799}{4}.$$

For $n \geq 1$

$$\begin{aligned}\frac{c_n}{R^{(n+1)}} &= -\frac{1}{2}\int_0^1 (2n+1)tP_n(t)dt \\ &= -\frac{1}{2}\int_0^1 \left[(n+1)P_{n+1}(t) + nP_{n-1}(t)\right] dt \\ &= -\frac{1}{2}\left[\frac{n+1}{2n+3}\left(P_n(0) - P_{n+2}(0)\right) + \frac{n}{2n-1}\left(P_{n-2}(0) - P_n(0)\right)\right]\end{aligned}$$

It follows that $c_{\text{odd}} = 0$ and for $n = 2k$,

$$c_{2k} = \left[\frac{(2k+1)P_{2k}(0)}{(8k+6)(4k-1)} + \frac{(2k+1)P_{2k+2}(0)}{8k+6} - \frac{kP_{2k-2}(0)}{4k-1}\right] R^{(2k+1)} =$$

The gravitational potential is

$$V(\rho,\phi) = \frac{799}{4}\,\frac{R}{\rho} + \sum_{k=1}^{\infty}\left[\frac{(2k+1)P_{2k}(0)}{(8k+6)(4k-1)} + \frac{(2k+1)P_{2k+2}(0)}{8k+6} - \frac{kP_{2k-2}(0)}{4k-1}\right]\left(\frac{R}{\rho}\right)^{(2k+1)} P_{2k}(\cos\phi)\,.$$

Exercise 25. The sun has a diameter of 1.4×10^6 km. If the temperature on the sun's surface is $5,000^0$ C, find the approximate temperature on the following planets.

Planet	Mean distance from sun (millions of kilometers)
Mercury	57.9
Venus	108.2
Earth	149.7
Mars	228.1
Jupiter	778.6
Saturn	1429.0
Uranus	2839.6
Neptune	4491.6
Pluto	5880.2