1. Exercises

In exercises 1 to 6, write a BVP for the small vertical vibrations of a homogeneous string. Assume that the wave's speed is c, the length of the string is L and satisfies the following conditions:

Exercise 1. Both ends of the string are fixed on the x-axis, the initial position of the string is given by $f(x) = \sin(5\pi x/L)$ and is released from rest (plucked string). Take $L = 20$ and $c = .5$.

$$\begin{array}{ll} u_{tt}(x,t) = 0.25\, u_{xx}(x,t) & 0 < x < 20\,, \quad t > 0 \\ u(x,0) = \dfrac{5\pi x}{20}, \quad u_t(x,0) = 0 & 0 < x < 20 \\ u(0,t) = 0, \quad u(20,t) = 0, & t > 0 \end{array}$$

Exercise 2. Same characteristics as in exercise 1 but this time while the string is siting horizontally at equilibrium, it is struck (at time 0) with an initial velocity given by $g(x) = x(x-1)\sin(\pi x/L)$ (struck string).

Exercise 3. Take $L = \pi$ and $c = 2$. Both ends of the string are fixed and initially the string has position given by $f(x) = \sin x \cos 2x$ and velocity given by $g(x) = -1$.

$$\begin{array}{ll} u_{tt}(x,t) = 4\, u_{xx}(x,t) & 0 < x < \pi\,, \quad t > 0 \\ u(x,0) = \sin x \cos(2x), \quad u_t(x,0) = -1 & 0 < x < \pi \\ u(0,t) = 0, \quad u(\pi,t) = 0, & t > 0 \end{array}$$

Exercise 4. Take $L = 2\pi$, $c = 2$. Suppose that the right end is fixed while the left end is allowed to move, vertically, in such a way that its vertical displacement at time t is $0.2\sin(t)$. The string starts its motion from rest at equilibrium position.

Exercise 5. Same string as in exercise 4. This time suppose that the left end is fixed while the right end is allowed to move vertically. At the right end, the displacement at any time t is equal to the slope of the tangent line. Suppose that the string is set into motion from equilibrium position by a constant velocity $g(x) = 1$.

$$\begin{array}{ll} u_{tt}(x,t) = 4\, u_{xx}(x,t) & 0 < x < 2\pi\,, \quad t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = 1 & 0 < x < 2\pi \\ u(0,t) = 0, \quad u(2\pi,t) = u_x(2\pi,t), & t > 0 \end{array}$$

Exercise 6. Same string as in exercise 4. This time suppose that the right end is fixed while the left end is allowed to move vertically. At the left end, the tangent line at any time t is horizontal. Suppose that the string is set into motion from rest with an initial position given by the function $f(x) = \sin 3x$.

Exercises 7 to 10 deal with the small vertical vibrations of a homogeneous membrane. You are asked to write the corresponding BVP.

Exercise 7. The membrane is a square with side π. The boundary is attached in the (x,y)-plane, the wave's speed is $c = 1$. Initially, the membrane position is given by $f(x,y) = \mathrm{e}^{-y}\sin x \sin 2y$. The membrane is released from rest.

$$\begin{array}{ll} u_{tt}(x,y,t) = u_{xx}(x,t) + u_{yy}(x,y,t) & 0 < x < \pi\,, \quad 0 < y < \pi \quad t > 0 \\ u(x,y,0) = \mathrm{e}^{-y}\sin x \sin 2y\,, \quad u_t(x,y,0) = 0 & 0 < x < \pi\,, \quad 0 < y < \pi \\ u(x,0,t) = 0, \quad u(x,\pi,t) = 0 & 0 < x < \pi\,, \quad t > 0 \\ u(0,y,t) = 0, \quad u(\pi,y,t) = 0 & 0 < y < \pi\,, \quad t > 0 \end{array}$$

Exercise 8. The membrane is a rectangle with boundary attached to the (x, y)-plane. At equilibrium position, the membrane occupies the rectangle $[-5,\ 5] \times [-3,\ 3]$. Suppose that the membrane is set into motion by striking its center square $[-1,\ 1]^2$ by a constant velocity of magnitude 1 (so at time $t = 0$, each point inside the small square has velocity 1 while the other points of the membrane outside the small square have velocity 0). Take $c = .5$

Exercise 9. The membrane is a circular disk with radius 10 and with boundary fixed on the (x, y)-plane (take $c = 2$ here). The initial velocity of the membrane is 0 and its initial position is given in polar coordinates by the (bump) function

$$f(r, \theta) = \begin{cases} e^{-r}(0.1 - r) & \text{if } 0 \le r \le 0.1 \\ 0 & \text{if } 0.1 < r \le 10. \end{cases}$$

The displacement $u(x, y, t)$ is defined for $x^2 + y^2 \le 10^2,\ \ t \ge 0$ and satisfies

$$\begin{array}{ll} u_{tt} = 4(u_{xx} + u_{yy}) & x^2 + y^2 < 100 \quad t > 0 \\ u(x, y, 0) = f(r, \theta)\,, \quad u_t(x, y, 0) = 0 & x = r\cos\theta, \quad y = r\sin\theta, \text{ with } 0 \le r < 10,\ 0 \le \theta \le 2\pi \\ u(x, y, t) = 0 & x^2 + y^2 = 100\,, \quad t > 0 \end{array}$$

Exercise 10. The membrane is a circular ring with inner radius 1 and outer radius 2 and $c = 1$. Suppose that the outer radius is fixed in the (x, y)-plane but the inner radius is allowed to move

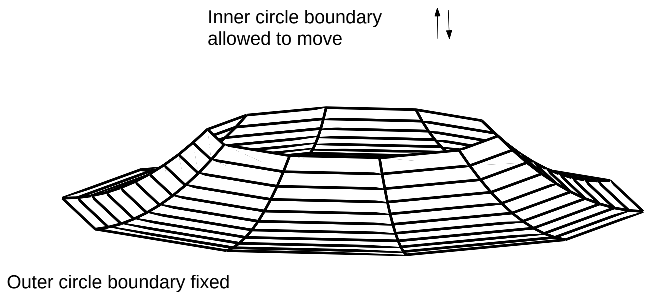


FIGURE 1. *Membrane in the shape of a ring*

vertically in such a way that at time t, each point of the inner radius has a displacement given by the function $e^{-0.001t}\sin t$. The initial position and velocity are 0 (see figure).

Exercise 11. The vertical displacements of a string were found to be given by the function $u(x, t) = \sin 3t \cos(x/2)$. What is the corresponding wave's speed c?

Let c be the wave speed. It follows from the wave equation that $c^2 = \dfrac{u_{tt}(x, t)}{u_{xx}(x, t)}$. For $u(x, t) = \sin 3t \cos(x/2)$ we get $c^2 = 36$ and $c = 6$.

Exercise 12. The vertical displacements of a membrane were found to be given by the function

$$u(x, y, t) = \cos(15t)\sin(3x)\cos(4y).$$

What is the corresponding wave's speed c?

Exercise 13. A string of length $L = 10$ and with fixed end on the horizontal axis is set into vertical motion by displacing it from its equilibrium position and then released from rest. The initial displacement is given by the function

$$f(x) = \begin{cases} x - 4 & \text{if } 4 < x < 5; \\ -x + 6 & \text{if } 5 < x < 6; \\ 0 & \text{if } 0 \le x \le 4 \text{ or } 6 \le x \le 10. \end{cases}$$

If $c = 2$, find the period of oscillations of the string. Use D'Alembert's method to find (and graph) the shape of the string at the following times $t = 1,\ 3,\ 5,\ 7,\ 9,\ 10$.

The period of oscillations is $T = \dfrac{2L}{c} = 10$. The solution $u(x,t) = \dfrac{\hat{F}(x-2t)+\hat{F}(x+2t)}{2}$ where $\hat{F}$ is the extension of f so that $\hat{F}$ is odd and with period $2L = 20$. The shape of the string at different times is given below

FIGURE 2. Shape of string at different times

Exercise 14. (D'Alembert's method for the struck string). Consider the BVP

$$\begin{array}{ll} u_{tt} = c^2 u_{xx} & 0 < x < L, \quad t > 0, \\ u(0,t) = u(L,t) = 0 & t > 0, \\ u(x,0) = 0 \quad u_t(x,0) = g(x) & 0 < x < L. \end{array}$$

We are going to construct a solution for this BVP. First, let g_{odd} be the odd extension of the function g to the interval $[-L,\ L]$. Second, let $\hat{g}$ be the periodic extension of g_{odd} to $\mathbb{R}$. Hence,

$\hat{g}$ has period $2L$. Third, let G be an antiderivative of $\hat{g}$. That is, $G'(s) = \hat{g}(s)$, for every $s \in \mathbb{R}$ and in particular, $G'(s) = g(s)$ if $0 \le s \le L$.

Verify that the function

$$u(x,t) = \frac{G(x+ct) - G(x-ct)}{2c}$$

solves the BVP.

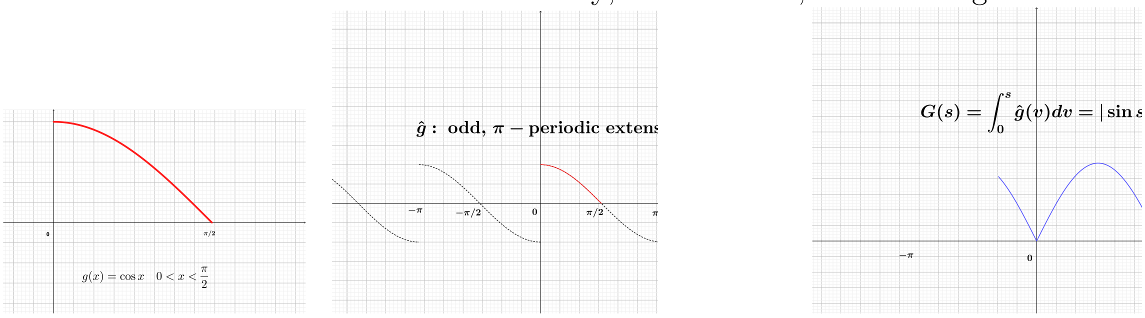
Exercise 15. Use the D'Alembert's method of exercise 14 to illustrate the shape of the struck string if

$$g(x) = \cos x \quad L = \pi/2, \quad c = .5$$

Find (graph) the shape of the string at the following times $t = 0,\ \pi/4,\ \pi/2,\ \pi,\ 2\pi$.

The solution $u(x,t)$ is given

FIGURE 3. Initial velocity; its extension; and its integral



$$u(x,t) = \frac{G(x+ct) - G(x-ct)}{2c} = |\sin(x+ct)| - |\sin(x-ct)|$$

Figure 4. Shape of string at different times