1. Exercises

All of the following exercises deal with the steady-state distribution of the temperature in either 2-dimensional plates or 3-dimensional regions. For each exercise, write the corresponding BVP.

Exercise 1. A 10×20 rectangular plate. The vertical sides are kept at constant temperatures. The left at 10 degrees and the right at 50 degrees. The horizontal sides are insulated. (Can you guess the solution for this problem?)

The steady-state temperature function $u = u(x, y)$ is defined in the rectangle $0 \leq x \leq 10$, $0 \leq y \leq 20$ and satisfies the BVP

$$\begin{array}{ll} u_{xx}(x,y) + u_{yy}(x,y) = 0 & 0 < x < 10\,, \quad 0 < y < 20 \\ u(0,y) = 10\,, \quad u(10,y) = 50 & 0 < y < 20 \\ u_y(x,0) = 0, \quad u_y(x,20) = 0, & 0 < x < 10 \end{array}$$

Since the vertical sides are maintained at constant temperatures and the horizontal sides insulated, this suggests that u is independent on y. The differential equation becomes $u_{xx} = 0$ and so $u(x, y) = Ax + B$. The boundary conditions $u(0, y) = 10$, $u(10, y) = 50$ give $A = 4$, $B = 10$. Hence $u(x, y) = 4x + 10$.

Exercise 2. A 10×20 rectangular plate with boundary conditions as in figure. At the lower side where there is poor insulation the normal derivative of the temperature is equal to 0.5 times the temperature.

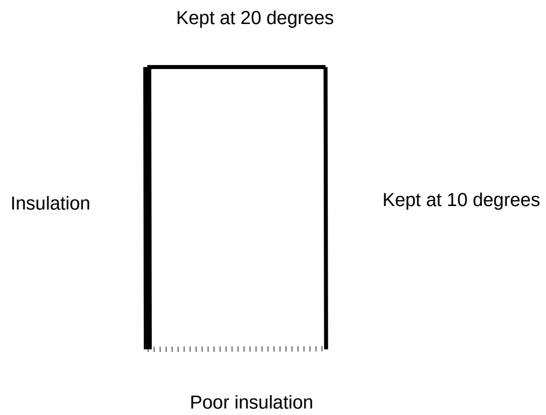

Figure 1. *Rectangular plate*

Exercise 3. A $1 \times 2 \times 3$ solid rectangular box. with boundary conditions as indicated in the

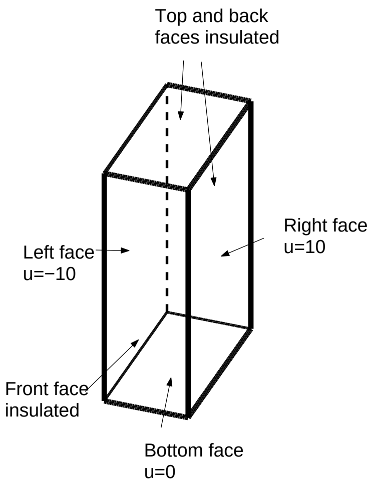

FIGURE 2. *Rectangular box*

figure.

The steady-state temperature function $u = u(x, y, z)$ is defined in the rectangular box $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 < z < 3$ and satisfies the BVP

$$\begin{array}{ll} u_{xx} + u_{yy} + u_{zz} = 0 & 0 < x < 1\,, \quad 0 < y < 2\,, \quad 0 < z < 3 \\ u_x(0, y, z) = 0, \quad u_x(1, y, z) = 0 & 0 < y < 2 \quad 0 < z < 3 \\ u(x, 0, z) = -10, \quad u(x, 2, z) = 10, & 0 < x < 1 \quad 0 < z < 3 \\ u(x, y, 0) = 0, \quad u_z(x, y, 3) = 0, & 0 < x < 1 \quad 0 < y < 2 \end{array}$$

Exercise 4. A plate in the shape of a quarter of a disk of radius 10 and with boundary conditions as indicated in the figure.

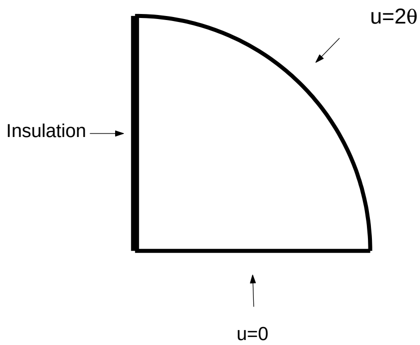

FIGURE 3. *A quarter of a disk shaped plate*

Exercise 5. A plate in the shape of half of a ring with inner radius 1 and outer radius b, (with $b > 1$), where the boundary conditions are as indicated in the figure.

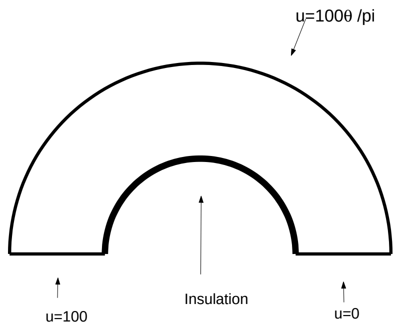

FIGURE 4. *A half-ring shaped plate*

Use polar coordinates (r, θ). The plate occupies the plane region given by $1 < r < b,\ \ 0 < \theta < \pi$. Recall the Laplace operator in polar coordinates is $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$. The steady-state temperature function $u = u(r, \theta)$ is satisfies the BVP

$$
\begin{array}{ll}
u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_r\theta\theta = 0 & 1 < r < b\,,\quad 0 < \theta < \pi \\[2ex]
u(r,0) = 0, \quad u(r,\pi) = 100 & 0 < r < b\, 0 < z < 3 \\
u(b,\theta) = 100\theta/\pi, \quad u_r(1,\theta) = 0, & 0 < \theta < \pi
\end{array}
$$

Exercise 6. A plate in the shape of a 45^0-sector of a ring with radii 1 and b, with $b > 1$, and where the boundary conditions are as indicated

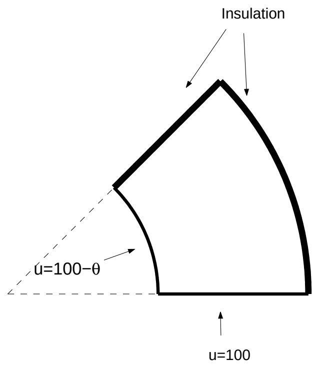

FIGURE 5. *A* 45^0 *sector of a ring shaped plate*

Exercise 7. A solid cylinder of radius 10 and height 20. The top surface is insulated, the bottom surface is kept at temperature 20 degrees and the lateral surface is kept at temperature 100 degrees.

The cylinder occupies the region given in cylindrical coordinates (r, θ, z) by $0 \le r < 10$, $0 \le \theta \le 2\pi$, $0 < z < 20$. The steady-state temperature function $u = u(r, \theta, z)$ satisfies the BVP

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_r\theta\theta + u_{zz} = 0 \qquad 1 \le r < 10\,, \quad 0 \le \theta \le 2\pi\,, \quad 0 < z < 20$$

$$\begin{array}{ll} u(r,\theta,0) = 20, \quad u_z(r,\theta,20) = 0 & 1 \le r < 10\,, \quad 0 \le \theta \le 2\pi\,, \\ u(10,\theta,z) = 100 & 0 \le \theta \le 2\pi\,, \quad 0 < z < 20 \end{array}$$

In addition u needs to be periodic function in θ with period 2π.

Exercise 8. A solid hollow cylinder (cylindrical shell) with radii 10 and 15 and with height 20. Assume that the inner lateral surface is insulated, the outer lateral surface is kept at temperature 100 degrees, the bottom surface is insulated and the top surface is kept at 50 degrees.

Exercise 9. A solid sphere with radius 10. The top hemisphere is kept at temperature 100 degrees and the lower hemisphere is kept at temperature 0 degrees.

Use spherical coordinates (ρ, θ, ϕ) so that the solid sphere occupies the region given $0 \le \rho \le 10$. The steady-state temperature function $u = u(\rho, \theta, \phi)$ satisfies the BVP

$$u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} + \frac{\cot\phi}{\rho^2}u_\phi + \frac{1}{\rho^2\sin^2\phi}u_{\theta\theta} = 0 \quad 0 < \rho < 10\,, \quad 0 \le \theta \le 2\pi\,, \quad 0 < \phi < \pi$$

$$\begin{array}{ll} u(10,\theta,\phi) = 100, & 0 \le \theta \le 2\pi\,, \quad 0 < \phi < \pi/2 \\ u(10,\theta,\phi) = 0, & 0 \le \theta \le 2\pi\,, \quad \pi/2 < \phi < \pi \end{array}$$

The function u needs to be 2π-periodic with respect to θ.

Exercise 10. A hollow solid sphere (spherical shell) with radii 1 and 5. The inner and outer surfaces are kept at temperatures 100 and 50 degrees, respectively. (Can you guess the temperature distribution?)