CLASSIFICATION AND PRINCIPLE OF SUPERPOSITION FOR SECOND ORDER LINEAR PDE

1. Exercises

In Exercises 1 to 5, classify the PDE as either linear or nonlinear and give its order.

Exercise 1. $u_{xx} + u_{yy} = \mathrm{e}^u$

Second order nonlinear.

Exercise 2. $3x^2 u_x - 2(\ln y)u_y + \dfrac{3}{2x-1}u = 5$

Exercise 3. $u_{xy} = 1$

Second order linear.

Exercise 4. $(u_x)^2 - 5u_y = 0$

Exercise 5. $u_{tt} - 2u_{xxxx} = \cos t$

Fourth order linear.

In Exercises 6 to 10, classify each second order linear PDE with constant coefficients as either elliptic, parabolic, or hyperbolic in the plane $\mathbb{R}^2$.

Exercise 6. $u_{xy} = 0$

Exercise 7. $u_{xx} - 2u_{xy} + 2u_{yy} + 5u_x - 12u_y + \sqrt{2}u = 0$

The discriminant is $\mathbb{D} = 2 - (-2)^2 = -2$ the pde is hyperbolic.

Exercise 8. $u_{xx} + 4u_{xy} + u_{yy} - 21u_y = \cos x$

Exercise 9. $2u_{xx} + 2\sqrt{2}u_{xy} + u_{yy} = \mathrm{e}^{xy}$

The discriminant is $\mathbb{D} = 2 - (2\sqrt{2})^2 = -6$ the pde is hyperbolic.

Exercise 10. $-u_{xx} + u_{xy} - u_{yy} + 3u = x^2$

In Exercises 11 to 13, find the regions in the (x, y)-plane where the second order PDE (with variable coefficients) is elliptic; parabolic; and hyperbolic.

Exercise 11. $x^2 u_{xx} + u_{yy} = 0$

The discriminant is $\mathbb{D}(x, y) = x^2$ the pde is elliptic for $x \neq 0$ and parabolic for $x = 0$.

Exercise 12. $\sqrt{x^2 + y^2}u_{xx} + 2u_{xy} + \sqrt{x^2 + y^2}u_{yy} + xu_x + yu_y = 0$

Exercise 13. $u_{xx} + 2yu_{xy} + xu_{yy} - \cos(xy)u_y = 1$

The discriminant is $\mathbb{D}(x, y) = x - 4y^2$ the pde is elliptic for $x > 4y^2$ and parabolic for $x = 4y^2$ and hyperbolic for $x < 4y^2$.

Exercise 14. Verify that the functions $(x+1)\mathrm{e}^{-t}$, $\mathrm{e}^{-2x}\sin t$ and xt are, respectively solutions of the nonhomogeneous equations

$$Hu = -\mathrm{e}^{-t}(x+1), \quad Hu = \mathrm{e}^{-2x}(4\sin t + \cos t), \quad \text{and} \quad Hu = x$$

where H is the 1-D heat operator $H = \dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}$

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Find a solution of the PDE

$$Hu = \sqrt{2}x + \pi e^{-2x}(4\sin t + \cos t) + e^{-t}(x+1)$$

Exercise 15. Verify that the functions $x\cos(x-t)$ and $\sin(x+t)+\cos(\sqrt{2}t)$ are, respectively solutions of the nonhomogeneous equations

$$\Box u = 2\sin(x-t), \qquad \text{and} \qquad \Box u = -2\cos(\sqrt{2}t)$$

where $\Box$ is the 1-D wave operator $\Box = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$

Find a solution of the PDE

$$\Box u = -\sin(x-t) + \pi\cos(\sqrt{2}t)$$

Let $v(x,t) = x\cos(x-t)$. Then $v_{tt} = -x\cos(x-t)$, and $v_{xx} = -2\sin(x-t) - x\cos(x-t)$. Hence

$$\Box v = (-x\cos(x-t)) - (-2\sin(x-t) - x\cos(x-t)) = 2\sin(x-t)\,.$$

Let $w(x,t) = \sin(x+t)+\cos(\sqrt{2}t)$. Then $w_{tt} = -(\sin(x+t)+2\cos(\sqrt{2}t))$, and $w_{xx} = -\sin(x+t)$. Hence

$$\Box w = -(\sin(x+t) + 2\cos(\sqrt{2}t)) + \sin(x+t) = -2\cos(\sqrt{2}t).$$

It follows from the superposition principle that the function

$$u(x,t) = -\frac{v(x,t) + \pi w(x,t)}{2} = -\frac{x\cos(x-t) + \pi\sin(x+t) + \pi\cos(\sqrt{2}t)}{2}$$

satisfies the equation $\Box u = -\sin(x-t) + \pi\cos(\sqrt{2}t)$.

Exercise 16. Verify that the functions r^2, $r^2\cos(2\theta)$ and $\sin(3\theta)$ are, respectively solutions of the PDEs

$$\Delta u = 4, \quad \Delta u = 0, \quad \text{and} \quad \Delta u = -\frac{9}{r^2}\sin(3\theta)$$

where Δ is the 2-D Laplace operator in polar coordinates $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}$

Find a solution of the PDE

$$\Delta u = -1 - \frac{\sin(3\theta)}{r^2}$$

In Exercises 17 to 20, decompose the given BVP into simpler BVPs in such a way that only one nonhomogeneous condition appears in each sub BVP.

Exercise 17.

$$\begin{array}{ll} u_t - ku_{xx} = \cos t & 0 < x < L, \quad t > 0 \\ u(x,0) = 3x & 0 < x < L \\ u(0,t) = 0, \quad u(L,t) = 20 & t > 0 \end{array}$$

Seek a solution u in the form $u(x,t) = v(x,t) + w^1(x,t) + w^2(x,t)$ where v, w^1, w^2 solve the following BVP:

$$\left\{\begin{array}{l} v_t - kv_{xx} = \cos t \\ v(x,0) = 0 \\ v(0,t) = 0, \quad v(L,t) = 0 \end{array}\right. \quad \left\{\begin{array}{l} w^1_t - kw^1_{xx} = 0 \\ w^1(x,0) = 3x \\ w^1(0,t) = 0, \quad w^1(L,t) = 0 \end{array}\right. \quad \left\{\begin{array}{l} w^2_t - kw^2_{xx} = 0 \\ w^2(x,0) = 0 \\ w^2(0,t) = 0, \quad w^2(L,t) = 20 \end{array}\right.$$

Exercise 18.

$$\begin{array}{ll} u_{tt} = 2u_{xx} + 2\sin x\cos t & 0 < x < \pi, \quad t > 0 \\ u(x,0) = \sin(3x), \quad u_t(x,0) = 1 & 0 < x < \pi \\ u(0,t) = \sin t, \quad u(\pi,t) = \cos t & t > 0 \end{array}$$

Exercise 19.

$$\begin{array}{ll} u_{xx}+u_{yy}=5\cos x\sin y & 0<x<\pi,\quad 0<y<\pi \\ u(x,0)=1,\quad u_y(x,\pi)=u(x,\pi) & 0<x<\pi \\ u(0,y)=-1,\quad u_x(\pi,y)=-3u(\pi,y) & 0<y<\pi \end{array}$$

We can find the solution u as $u=u^1+u^2+u^3$ with

$$\begin{cases} u^1_{xx}+u^1_{yy}=5\cos x\sin y \\ u^1(x,0)=0,\quad u^1_y(x,\pi)=u^1(x,\pi) \\ u^1(0,y)=0,\quad u^1_x(\pi,y)=-3u^1(\pi,y) \end{cases}$$

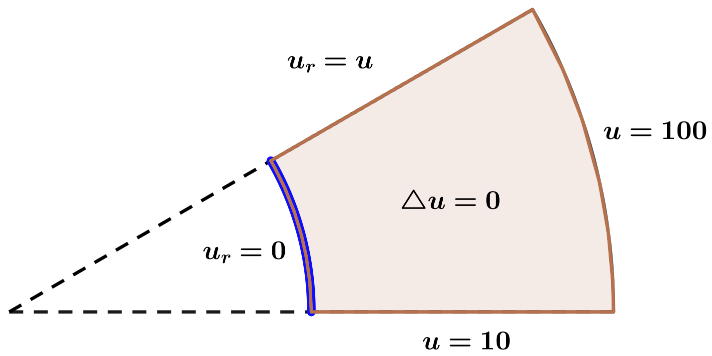
$$\begin{cases} u^2_{xx}+u^2_{yy}=0 \\ u^2(x,0)=1,\quad u^2_y(x,\pi)=u^2(x,\pi) \\ u^2(0,y)=0,\quad u^2_x(\pi,y)=-3u^2(\pi,y) \end{cases} \qquad \begin{cases} u^3_{xx}+u^3_{yy}=0 \\ u^3(x,0)=0,\quad u^3_y(x,\pi)=u^3(x,\pi) \\ u^3(0,y)=-1,\quad u^3_x(\pi,y)=-3u^3(\pi,y) \end{cases}$$

Exercise 20.

$$\begin{array}{ll} u_{rr}+\dfrac{1}{r}u_r+\dfrac{1}{r^2}u_{\theta\theta}=0 & 0<r<1,\quad 0<\theta<\pi \\ u(r,0)=10,\quad u(r,\pi)=20, & 0<r<1 \\ u_r(1,\theta)=5u(1,\theta) & 0<\theta<\pi \end{array}$$

Exercise 21. Write the BVP for the steady-state temperature in a plate in the form of 30^0-sector of a ring with radii 1 and 2. One of the radial edges is kept at temperature 10^0C and on the other radial edge, the gradient of the temperature is numerically equal to the temperature. The outer circular edge is kept at temperature 100^0C, while the inner circular edge is insulated.

Decompose the BVP into Sub-BVPs that contain only one nonhomogeneous condition.



The solution $u(r,\theta)$ defined for $1<r<2,\quad 0<\theta<\pi/6$ can be written as $u(r,\theta)=v(r,\theta)+w(r,\theta)$ where v and w satisfy the following BVPs

$$\begin{cases} v_{rr}+\dfrac{1}{r}v_r+\dfrac{1}{r^2}v_{\theta\theta}=0 \\ v(2,\theta)=100\quad v_r(1,\theta)=0 \\ v(r,0)=0 \end{cases} \qquad \begin{cases} w_{rr}+\dfrac{1}{r}w_r+\dfrac{1}{r^2}w_{\theta\theta}=0 \\ w(2,\theta)=0\quad w_r(1,\theta)=0 \\ w(r,0)=10 \end{cases}$$