THE METHOD OF SEPARATION OF VARIABLES

Exercises

Use the method of separation of variables to solve the following boundary value problems.

Exercise 1.

$$\begin{array}{ll} u_t(x,t) = 3u_{xx}(x,t) & 0 < x < 2\pi\,, \quad t > 0 \\ u(0,t) = u(2\pi,t) = 0 & t > 0 \\ u(x,0) = \sin\dfrac{x}{2} - \sin(3x) & 0 < x < 2\pi \end{array}$$

- Homogeneous and nonhomogeneous parts of the BVP are:

$$\text{HP:} \quad \begin{cases} u_t(x,t) = 3u_{xx}(x,t) \\ u(0,t) = u(2\pi,t) = 0 \end{cases} \qquad \text{NHP:} \quad u(x,0) = \sin\frac{x}{2} - \sin(3x)$$

- ODE problems for solutions with separated variables of HP. Suppose $u(x,t) = X(x)T(t)$ is a nontrivial solution of HP. Then $X(x)$ and $T(t)$ solve the following ODE problems:

$$\begin{cases} X''(t) + \lambda X(t) = 0 \\ X(0) = X(2\pi) = 0 \end{cases} \qquad T'(t) + 3\lambda T(t) = 0$$

  where λ is the separation constant.
- Eigenvalues and eigenfunctions of the X-problem.

  Eigenvalues: $\lambda_n = \dfrac{n^2}{4}$; eigenfunctions $X_n(x) = \sin\dfrac{nx}{2}$, with $n \in \mathbb{Z}^+$.
- For each eigenvalue λ_n, the corresponding T-problem has solutions generated by $T_n(t) = \mathrm{e}^{-3n^2t/4}$.
- Solutions with separated variables of HP:

$$u_n(x,t) = \mathrm{e}^{-3n^2t/4}\sin\frac{nx}{2} \qquad n \in \mathbb{Z}^+$$

- Superposition gives more general solutions of HP

$$u(x,t) = C_1\mathrm{e}^{-3t/4}\sin\frac{x}{2} + C_2\mathrm{e}^{-3t}\sin x + \cdots + C_N\mathrm{e}^{-3N^2t/4}\sin\frac{Nx}{2}$$

- Use NHP to find the constants $C_1, \cdots, C_N$ so that the solution solve the original BVP (HP and NHP).

$$u(x,0) = \sin\frac{x}{2} - \sin(3x) = C_1\sin\frac{x}{2} + C_2\sin x + \cdots + C_N\sin\frac{Nx}{2}$$

  By identifying coefficients we find $C_j = 0$ for $j \neq 1, 6$, $C_1 = 1$, and $C_6 = -1$.
- The solution of the BVP is

$$u(x,t) = \mathrm{e}^{-3t/4}\sin\frac{x}{2} - \mathrm{e}^{-27t}\sin(3x)$$

Exercise 2.

$$\begin{array}{ll} u_t(x,t) = u_{xx}(x,t) & 0 < x < 2\pi\,, \quad t > 0 \\ u_x(0,t) = u_x(2\pi,t) = 0 & t > 0 \\ u(x,0) = 100 - 3\cos\dfrac{5x}{4} & 0 < x < 2\pi \end{array}$$

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Exercise 3.

$$\begin{array}{ll} u_t(x,t) = 2u_{xx}(x,t) & 0 < x < \pi\,, \quad t > 0 \\ u(0,t) = u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = 50\sin\frac{x}{2} - 25\sin\frac{7x}{2} + 10\sin\frac{11x}{2} & 0 < x < \pi \end{array}$$

- Homogeneous and nonhomogeneous parts of the BVP are:

HP: $\begin{cases} u_t(x,t) = 2u_{xx}(x,t) \\ u(0,t) = 0, \; u_x(\pi,t) = 0 \end{cases}$ NHP: $u(x,0) = 50\sin\frac{x}{2} - 25\sin\frac{7x}{2} + 10\sin\frac{11x}{2}$

- ODE problems for solutions with separated variables of HP. Suppose $u(x,t) = X(x)T(t)$ is a nontrivial solution of HP. Then $X(x)$ and $T(t)$ solve the following ODE problems:

$$\begin{cases} X''(t) + \lambda X(t) = 0 \\ X(0) = 0, \; X'(\pi) = 0 \end{cases} \qquad T'(t) + 2\lambda T(t) = 0$$

  where λ is the separation constant.
- Eigenvalues and eigenfunctions of the X-problem.
  Eigenvalues: $\lambda_n = \frac{(2n+1)^2}{4}$; eigenfunctions $X_n(x) = \sin\frac{(2n+1)x}{2}$, with $n = 0, 1, 2, 3, \cdots$.
- For each eigenvalue λ_n, the corresponding T-problem has solutions generated by $T_n(t) = \mathrm{e}^{-(2n+1)^2 t/2}$.
- Solutions with separated variables of HP:

$$u_n(x,t) = \mathrm{e}^{-(2n+1)^2 t/2}\sin\frac{nx}{2} \qquad n \in \mathbb{Z}^+ \cup \{0\}$$

- Superposition gives more general solutions of HP

$$u(x,t) = \sum_{j=0}^{N} C_j \mathrm{e}^{-(2j+1)^2 t/2}\sin\frac{(2j+1)x}{2}$$

- Use NHP to find the constants $C_1, \cdots, C_N$ so that the solution solve the original BVP (HP and NHP).

$$u(x,0) = 50\sin\frac{x}{2} - 25\sin\frac{7x}{2} + 10\sin\frac{11x}{2} = \sum_{j=0}^{N} C_j \sin\frac{(2j+1)x}{2}$$

  By identifying coefficients we find $C_j = 0$ for $j \neq 0, 3, 5$; $C_0 = 50$, $C_3 = -25$, and $C_5 = 10$.
- The solution of the BVP is

$$u(x,t) = 50\mathrm{e}^{-t/2}\sin\frac{x}{2} - 25\mathrm{e}^{-49t/2}\sin\frac{7x}{2} + 10\mathrm{e}^{-121t/2}\sin\frac{11x}{2}$$

Exercise 4.

$$\begin{array}{ll} u_{tt}(x,t) = \sqrt{2}u_{xx}(x,t) & 0 < x < 3\pi\,, \quad t > 0 \\ u(0,t) = u(3\pi,t) = 0 & t > 0 \\ u_t(x,0) = 0 & 0 < x < 3\pi \\ u(x,0) = f(x) & 0 < x < 3\pi \end{array}$$

where

$$f(x) = \sin\frac{x}{3} - \frac{1}{2}\sin(2x) + \frac{1}{5}\sin\frac{7x}{3}$$

Exercise 5.

$$\begin{array}{ll} u_{tt}(x,t) = 4u_{xx}(x,t) & 0 < x < 3\pi\,, \quad t > 0 \\ u(0,t) = u(3\pi,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < 3\pi \\ u_t(x,0) = g(x) & 0 < x < 3\pi \end{array}$$

where

$$g(x) = \sin\frac{2x}{3} + \frac{1}{2}\sin\frac{5x}{3}$$

- Homogeneous and nonhomogeneous parts of the BVP are:

$$\text{HP:} \quad \begin{cases} u_{tt}(x,t) = 4u_{xx}(x,t) \\ u(0,t) = u(3\pi,t) = 0 \\ u(x,0) = 0 \end{cases} \qquad \text{NHP:} \quad u_t(x,0) = g(x)$$

- ODE problems for solutions with separated variables of HP. Suppose $u(x,t) = X(x)T(t)$ is a nontrivial solution of HP. Then $X(x)$ and $T(t)$ solve the following ODE problems:

$$\begin{cases} X''(t) + \lambda X(t) = 0 \\ X(0) = 0,\ X(3\pi) = 0 \end{cases} \qquad \begin{cases} T'(t) + 4\lambda T(t) = 0 \\ T(0) = 0 \end{cases}$$

  where λ is the separation constant.
- Eigenvalues and eigenfunctions of the X-problem.
  Eigenvalues: $\lambda_j = \dfrac{j^2}{9}$; eigenfunctions $X_j(x) = \sin\dfrac{jx}{3}$, with $j = 1,\ 2,\ 3,\ \cdots$.
- For each eigenvalue λ_j, the corresponding T-problem has solutions generated by $T_n(t) = \sin\dfrac{2jt}{3}$.
- Solutions with separated variables of HP:

$$u_j(x,t) = \sin\frac{2jt}{3}\sin\frac{jx}{3} \qquad j \in \mathbb{Z}^+$$

- Superposition gives more general solutions of HP

$$u(x,t) = \sum_{j=0}^{N} C_j \sin\frac{2jt}{3}\sin\frac{jx}{3}$$

- Use NHP to find the constants $C_1, \cdots, C_N$ so that the solution solve the original BVP (HP and NHP). We have

$$u_t(x,t) = \sum_{j=0}^{N} \frac{2j}{3} C_j \cos\frac{2jt}{3}\sin\frac{jx}{3}$$

  and

$$u_t(x,0) = \sum_{j=0}^{N} \frac{2j}{3} C_j \sin\frac{jx}{3} = \sin\frac{2x}{3} + \frac{1}{2}\sin\frac{5x}{3}$$

  By identifying coefficients we find $C_j = 0$ for $j \neq 2,\ 5$; $C_2 = 3/4$ and $C_5 = 3/20$.
- The solution of the BVP is

$$u(x,t) = \frac{3}{4}\sin\frac{4t}{3}\sin\frac{2x}{3} + \frac{3}{20}\sin\frac{10t}{3}\sin\frac{5x}{3}$$

Exercise 6.

$$\begin{array}{ll} u_{tt}(x,t) = u_{xx}(x,t) & 0 < x < 1\,, \quad t > 0 \\ u(0,t) = u_x(1,t) = 0 & t > 0 \\ u(x,0) = \sin\dfrac{\pi x}{2} & 0 < x < 1 \\ u_t(x,0) = -\sin\dfrac{7\pi x}{2} & 0 < x < 1 \end{array}$$

Exercise 7.

$$\begin{array}{ll} u_{tt}(x,t) = u_{xx}(x,t) & 0 < x < 2\,, \quad t > 0 \\ u_x(0,t) = u_x(2,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < 2 \\ u_t(x,0) = 1 - \dfrac{1}{2}\cos\dfrac{5\pi x}{2} & 0 < x < 2 \end{array}$$

- Homogeneous and nonhomogeneous parts of the BVP are:

$$\text{HP:} \quad \begin{cases} u_{tt}(x,t) = u_{xx}(x,t) \\ u_x(0,t) = u_x(2,t) = 0 \\ u(x,0) = 0 \end{cases} \qquad \text{NHP:} \quad u_t(x,0) = 1 - \frac{1}{2}\cos\frac{5\pi x}{2}$$

- ODE problems for solutions with separated variables of HP. Suppose $u(x,t) = X(x)T(t)$ is a nontrivial solution of HP. Then $X(x)$ and $T(t)$ solve the following ODE problems:

$$\begin{cases} X''(t) + \lambda X(t) = 0 \\ X'(0) = 0, \;\; X'(2) = 0 \end{cases} \qquad \begin{cases} T'(t) + \lambda T(t) = 0 \\ T(0) = 0 \end{cases}$$

  where λ is the separation constant.
- Eigenvalues and eigenfunctions of the X-problem.

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2, \quad X_n(x) = \cos\frac{n\pi x}{2}, \quad n = 0, 1,\; 2,\; 3,\; \cdots$$

  Note that $\lambda_0 = 0$ is an eigenvalue with eigenfunction $X_0(x) = 1$.
- For each eigenvalue λ_n, the corresponding T-problem has solutions generated by $T_n(t) = \sin\dfrac{n\pi t}{2}$ for $n = 1,\; 2,\; 3 \cdots$ and $T_0(t) = t$ for $n = 0$.
- Solutions with separated variables of HP:

$$u_0(x,t) = t, \qquad u_n(x,t) = \sin\frac{n\pi t}{2}\cos\frac{n\pi x}{2} \qquad n \in \mathbb{Z}^+$$

- Superposition gives more general solutions of HP

$$u(x,t) = A_0 t + \sum_{j=1}^{N} C_j \sin\frac{j\pi t}{2}\cos\frac{j\pi x}{2}$$

- Use NHP to find the constants $C_1, \cdots, C_N$ so that the solution solve the original BVP (HP and NHP). We have

$$u_t(x,t) = A_0 + \sum_{j=1}^{N} \frac{j\pi}{2} A_j \cos\frac{j\pi t}{3}\cos\frac{j\pi x}{2}$$

  and

$$u_t(x,0) = A_0 + \sum_{j=1}^{N} \frac{j\pi}{2} A_j \cos\frac{j\pi x}{2} = 1 - \frac{1}{2}\cos\frac{5\pi x}{2}$$

  By identifying coefficients we find $A_j = 0$ for $j \neq 0,\; 5;\;\; A_0 = 1$ and $A_5 = -1/5\pi$.

- The solution of the BVP is

$$u(x,t) = t - \frac{1}{5\pi}\sin\frac{5\pi t}{2}\cos\frac{5\pi x}{2}$$

Exercise 8.

$$\begin{array}{ll} \Delta u(x,y) = 0 & 0 < x < 1\,, \quad 0 < y < 2 \\ u(0,y) = u(1,y) = 0 & 0 < y < 2 \\ u(x,0) = \sin(2\pi x), \quad u(x,2) = \sin(3\pi x) & 0 < x < 1 \end{array}$$

Exercise 9.

$$\begin{array}{ll} \Delta u(x,y) = 0 & 0 < x < 1\,, \quad 0 < y < 2 \\ u(0,y) = 1 - \cos\frac{\pi y}{2}, \quad u(1,y) = 3\cos\frac{5\pi y}{2} & 0 < y < 2 \\ u_y(x,0) = u_y(x,2) = 0 & 0 < x < 1 \end{array}$$

- Homogeneous and nonhomogeneous parts of the BVP are:

$$\text{HP:} \ \begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0 \\ u_y(x,0) = u_y(x,2) = 0 \end{cases} \qquad \text{NHP:} \ u(0,y) = 1 - \cos\frac{\pi y}{2}, \quad u(1,y) = 3\cos\frac{5\pi y}{2}$$

- ODE problems for solutions with separated variables of HP. Suppose $u(x,y) = X(x)Y(y)$ is a nontrivial solution of HP. Then $X(x)$ and $Y(y)$ solve the following ODE problems:

$$\begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y'(0) = 0, \ Y'(2) = 0 \end{cases} \qquad X''(x) - \lambda X(x) = 0$$

  where λ is the separation constant.
- Eigenvalues and eigenfunctions of the Y-problem.

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2, \quad Y_n(y) = \cos\frac{n\pi y}{2}, \quad n = 0, 1, \ 2, \ 3, \ \cdots$$

  Note that $\lambda_0 = 0$ is an eigenvalue with eigenfunction $Y_0(y) = 1$.
- For each eigenvalue λ_n ($n \geq 1$), the corresponding X-problem has solutions generated by $X_{1,n}(x) = \sinh\dfrac{n\pi x}{2}$ and $X_{2,n}(x) = \cosh\dfrac{n\pi x}{2}$ (note that instead of using hyperbolic trigonometric functions, we could use $e^{\frac{n\pi x}{2}}$ and $e^{-\frac{n\pi x}{2}}$ For the eigenvalue $\lambda_0 = 0$, the corresponding X-problem has solutions generated by $X_{0,1}(x) = 1$ and $X_{0,1}(x) = x$
- Solutions with separated variables of HP:

$$u_{0,1}(x,y) = 1, \quad u_{0,2}(x,y) = x, \quad \text{for } \lambda = 0$$

$$\begin{cases} u_{n,1}(x,y) & = \sinh\frac{n\pi x}{2}\cos\frac{n\pi y}{2}, \\ u_{n,2}(x,y) & = \cosh\frac{n\pi x}{2}\cos\frac{n\pi y}{2} \end{cases} \qquad \text{for } \lambda_n = \left(\tfrac{n\pi}{2}\right)^2 \quad n \in \mathbb{Z}^+$$

- Superposition gives more general solutions of HP

$$u(x,y) = A_0 + A_1 x + \sum_{j=1}^{N}\left(C_j\sinh\frac{j\pi x}{2} + D_j\cosh\frac{j\pi x}{2}\right)\cos\frac{j\pi y}{2}$$

- Use NHP to find the constants so that the solution solve the original BVP (HP and NHP). We get

$$u(0,y) = 1 - \cos\frac{\pi y}{2} = A_0 + \sum_{j=1}^{N} D_j \cosh\frac{j\pi x}{2}\cos\frac{j\pi y}{2}$$

$$u(1,y) = 3\cos\frac{\pi y}{2} = A_0 + A_1 + \sum_{j=1}^{N}\left(C_j \sinh\frac{j\pi}{2} + D_j\cosh\frac{j\pi}{2}\right)\cos\frac{j\pi y}{2}$$

and deduce $A_0 = 1$, $A_1 = -1$, $C_j = 0$ for $j \neq 1,\ 5$, $D_j = 0$ for $j \neq 1$, $D_1 = -1$, $C_1 = \cosh(\pi/2)/\sinh(\pi/2)$, and $C_5 = 3/\sinh(5\pi/2)$

- The solution of the BVP (after simplification) is

$$u(x,y) = 1 - x + \frac{\sinh\frac{\pi(x-1)}{2}}{\sinh\frac{\pi}{2}}\cos\frac{\pi y}{2} + 3\frac{\sinh\frac{5\pi x}{2}}{\sinh\frac{5\pi}{2}}\cos\frac{5\pi y}{2}$$

Exercise 10. (In polar coordinates)

$$\begin{array}{ll} \Delta u(r,\theta) = 0 & 0 < r < 2\,,\quad 0 \le \theta \le 2\pi \\ u(2,\theta) = 1 + \cos(3\theta) - 2\sin(5\theta) & 0 \le \theta \le 2\pi \end{array}$$

Exercise 11. (In polar coordinates)

$$\begin{array}{ll} \Delta u(r,\theta) = 0 & 1 < r < 2\,,\quad 0 \le \theta \le 2\pi \\ u(1,\theta) = \sin(3\theta) & 0 \le \theta \le 2\pi \\ u_r(2,\theta) = \cos(5\theta) & 0 \le \theta \le 2\pi \end{array}$$

- Homogeneous and nonhomogeneous parts of the BVP are:

$$\text{HP:}\quad \begin{cases} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) = 0 \\ u(r,0) = u(r,2\pi), \quad u_\theta(r,0) = u_\theta(r,2\pi) \end{cases} \qquad \text{NHP:}\quad \begin{cases} u(1,\theta) = \sin(3\theta) \\ u_r(2,\theta) = \cos(5\theta) \end{cases}$$

- ODE problems for solutions with separated variables of HP. Suppose $u(r,\theta) = R(r)\Theta(\theta)$ is a nontrivial solution of HP. Then $R(r)$ and $\Theta(\theta)$ solve the following ODE problems:

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \\ \Theta(0) = \Theta(2\pi) \quad \Theta'(0) = \Theta'(2\pi) \end{cases} \qquad r^2R''(r) + rR'(r) - \lambda R(r) = 0$$

where λ is the separation constant. Note that the R-equation is Cauchy-Euler.

- Eigenvalues and eigenfunctions of the Θ-problem.

Eigenvalue	Eigenfunction
$\lambda_0 = 0$	$\Theta_0(\theta) = 1$
$\lambda_n = n^2$	$\begin{cases} \Theta_{1,n}(\theta) = \cos(n\theta) \\ \Theta_{2,n}(\theta) = \sin(n\theta) \end{cases}$

Note that for $n \in \mathbb{Z}^+$ there are two independent eigenfunctions ($\cos(n\theta)$ and $\cos(n\theta)$) corresponding to the eigenfunction $\lambda_n = n^2$.

- For each eigenvalue, the corresponding R-problem has solutions generated by

$$\lambda_0 = 0 \quad \begin{cases} R_{1,0}(r) = 1 \\ R_{2,0}(r) = \ln r \end{cases}$$

$$\lambda_n = n^2 \quad \begin{cases} R_{1,n}(r) = r^n \\ R_{2,n}(r) = r^{-n} \end{cases}$$

- Solutions with separated variables of HP:

$$\begin{array}{ll} \lambda_0 = 0 & 1,\ \ln r \\ \lambda_n = n^2 & r^n \cos(n\theta), \quad r^{-n}\cos(n\theta), \quad r^n \sin(n\theta), \quad r^{-n}\sin(n\theta) \end{array}$$

- Superposition gives more general solutions of HP

$$u(r,\theta) = P_0 + P_1 \ln r + \sum_{j=1}^{N} \left(A_j r^j + \frac{B_j}{r^j}\right)\cos(j\theta) + \left(C_j r^j + \frac{D_j}{r^j}\right)\sin(j\theta)$$

- Use NHP to find the constants so that the solution solve the original BVP (HP and NHP). First we compute u_r:

$$u_r(r,\theta) = \frac{P_1}{r} + \sum_{j=1}^{N} \left(jA_j r^{j-1} - \frac{jB_j}{r^{j+1}}\right)\cos(j\theta) + \left(jC_j r^{j-1} - \frac{jD_j}{r^{j+1}}\right)\sin(j\theta)$$

We have

$u(1,\theta) = \sin(3\theta) = P_0 + \sum_{j=1}^{N} (A_j + B_j)\cos(j\theta) + (C_j + D_j)\sin(j\theta)$

$$u_r(2,\theta) = \cos(5\theta) = \frac{P_1}{2} + \sum_{j=1}^{N} \left(jA_j 2^{j-1} - \frac{jB_j}{2^{j+1}}\right)\cos(j\theta) + \left(jC_j 2^{j-1} - \frac{jD_j}{2^{j+1}}\right)\sin(j\theta)$$

and deduce $P_0 = P_1 = 0$; $A_j = B_j = 0$ for $j \neq 5$; $C_j = D_j = 0$ for $j \neq 3$; $A_5 = \dfrac{64}{5125}$, $B_5 = -\dfrac{64}{5125}$, $C_3 = \dfrac{1}{65}$, $\dfrac{64}{65}$.

- The solution of the BVP is

$$u(r,\theta) = \frac{64}{6125}\left(r^5 - \frac{1}{r^5}\right)\cos(5\theta) + \frac{1}{65}\left(r^3 + \frac{64}{r^3}\right)\sin(3\theta)$$

Exercise 12.

$$\begin{array}{ll} u_t(x,y,t) = \Delta u(x,y,t) & 0 < x < \pi\,,\ 0 < y < 2\pi\,,\ t > 0 \\ u(x,0,t) = u(x,2\pi,t) = 0 & 0 < x < \pi\,,\ t > 0 \\ u(0,y,t) = u(\pi,y,t) = 0 & 0 < y < 2\pi\,,\ t > 0 \\ u(x,y,0) = \sin x \sin \dfrac{3y}{2} & 0 < x < \pi\,,\ 0 < y < 2\pi \end{array}$$

Exercise 13.

$$\begin{array}{ll} u_t(x,y,t) = \Delta u(x,y,t) & 0 < x < \pi\,,\ 0 < y < 2\pi\,,\ t > 0 \\ u(x,0,t) = u(x,2\pi,t) = 0 & 0 < x < \pi\,,\ t > 0 \\ u_x(0,y,t) = u_x(\pi,y,t) = 0 & 0 < y < 2\pi\,,\ t > 0 \\ u(x,y,0) = 2\sin 5x \sin \dfrac{3y}{2} & 0 < x < \pi\,,\ 0 < y < 2\pi \end{array}$$

- Homogeneous and nonhomogeneous parts of the BVP are:

$$\text{HP:} \quad \begin{cases} u_t(x,y,t) = \Delta u(x,y,t) \\ u(x,0,t) = u(x,2\pi,t) = 0 \\ u_x(0,y,t) = u_x(\pi,y,t) = 0 \end{cases} \qquad \text{NHP:} \quad u(x,y,0) = 2\sin 5x \sin\frac{3y}{2}$$

- ODE problems for solutions with separated variables of HP. Suppose $u(x,y,t) = X(x)Y(y)T(t)$ is a nontrivial solution of HP. Then $X(x)$, $Y(y)$, and $T(t)$ solve the following ODE problems:

$$\begin{cases} X"(x) + \alpha X(x) = 0 \\ X'(0) = X'(\pi) = 0 \end{cases} \qquad \begin{cases} Y"(y) + \beta Y(y) = 0 \\ Y(0) = Y(2\pi) = 0 \end{cases} \qquad T'(t) + \lambda T(t) = 0$$

where α, β, λ are separation constants with $\lambda = \alpha + \beta$.
- Eigenvalues and eigenfunctions of the X and Y problems.

	X-Problem	Y-Problem
Eigenvalues	$\alpha_n = n^2$ $n = 0, 1, 2, 3, \cdots$	$\beta_m = \dfrac{m^2}{4}$ $m = 1, 2, 3, \cdots$
Eigenfunctions	$X_n(x) = \cos(nx)$	$Y_m(y) = \sin\dfrac{my}{2}$

- For each pair of eigenvalues α_n, β_m, the corresponding T-problem has solutions generated by

$$T_{n,m}(t) = \mathrm{e}^{-(\alpha_n+\beta_m)t}$$

- Solutions with separated variables of HP:

$$u_{n,m}(x,y,t) = \mathrm{e}^{-(\alpha_n+\beta_m)t}\cos(nx)\sin\frac{my}{2} \qquad m = 1, 2, 3, \cdots \quad n = 0, 1, 2, \cdots$$

- Superposition gives more general solutions of HP

$$u(x,y,t) = \sum_{n=0}^{N}\sum_{m=1}^{M} A_{n,m}\mathrm{e}^{-(\alpha_n+\beta_m)t}\cos(nx)\sin\frac{my}{2}$$

- Use NHP to find the constants so that the solution solve the original BVP (HP and NHP).

$$u(x,y,0) = 2\sin 5x \sin\frac{3y}{2} = \sum_{n=0}^{N}\sum_{m=1}^{M} A_{n,m}\cos(nx)\sin\frac{my}{2}$$

We obtain $A_{n,m} = 0$ for $(n,m) \neq (5,3)$ and $A_{5,3} = 2$. The solution of the BVP is

$$u(x,y,t) = 2\mathrm{e}^{-\frac{109\,t}{4}}\cos(5x)\sin\frac{3y}{2}$$

Exercise 14.

$$\begin{array}{ll} u_{tt}(x,y,t) = 2\Delta u(x,y,t) & 0 < x < \pi\,,\ 0 < y < 2\pi\,,\ t > 0 \\ u_y(x,0,t) = u_y(x,2\pi,t) = 0 & 0 < x < \pi\,,\ t > 0 \\ u(0,y,t) = u(\pi,y,t) = 0 & 0 < y < 2\pi\,,\ t > 0 \\ u(x,y,0) = \sin 3x \cos\dfrac{5y}{2} & 0 < x < \pi\,,\ 0 < y < 2\pi \\ u_t(x,y,0) = 0 & 0 < x < \pi\,,\ 0 < y < 2\pi \end{array}$$

In exercises 15 to 17 find all solutions with separated variables.

Exercise 15.

$$\begin{array}{ll} u_t(x,t) = 3u_{xx}(x,t) & 0 < x < 2\,,\quad t > 0 \\ u(0,t) = 0 & t > 0 \\ u(2,t) = -u_x(2,t) & t > 0 \end{array}$$

If $u(x,t) = X(x)T(t)$ solves the BVP, then X and T satisfy the following ODE problems

$$\begin{cases} X"(x) + \lambda X(x) = 0 \\ X(0) = 0, \quad X(2) = -X'(2) \end{cases} \qquad T'(t) + 3\lambda T(t) = 0$$

To find the eigenvalues and eigenfunctions of the X-problem, we consider 3 cases

- *Case* $\lambda < 0$: Set $\lambda = -\nu^2$ with $\nu > 0$. Then $X(x) = C_1 e^{\nu x} + C_2 e^{-\nu x}$. Then end point condition $X(0) = 0$ leads to $C_2 = -C_1$ and then if $C_1 \neq 0$ we get$e^{4\nu} = \dfrac{1-\nu}{1+\nu}$. This last equation has no positive solutionν since $e^{4\nu} > 1$ and for $\nu > 0$ and $\dfrac{1-\nu}{1+\nu} < 1$. Thus $\lambda < 0$ cannot be an eigenvalue.
- *Case* $\lambda = 0$: In this case the only solution of the X-problem is trivial.
- *Case* $\lambda > 0$: Set $\lambda = \nu^2$ with $\nu > 0$. The general solution of the ODE is $X(x) = C_1 \cos(\nu x) + C_2 \sin(\nu x)$. The first boundary condition gives $C_1 = 0$ and the second leads to $C_2 \tan(2\nu) = -C_2 \nu$. In order to get nontrivial solution $X(x)$, it is therefore necessary for ν to satisfy the equation $\tan(2\nu) = -\nu$.

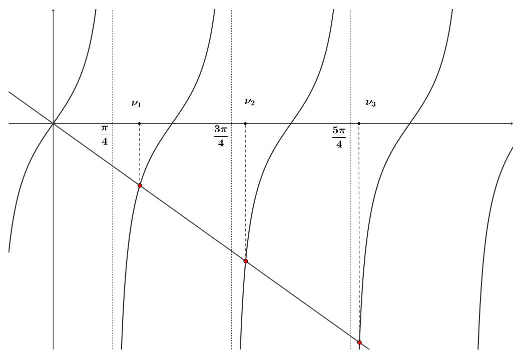

FIGURE 1. *Solutions of* $\tan(2\nu) = -\nu$

Equation $\tan(2\nu) = -\nu$ has infinitely many solutions $0 < \nu_1 < \nu_2 < \nu_3 < \cdots$ with $\nu_n \in \left(\frac{(2n-1)\pi}{4}, \frac{2n\pi}{4}\right)$.

For each eigenvalue $\lambda_n = \nu_n^2$, the solutions of the corresponding T-problem are generated by $T_n(t) = e^{-3\nu_n^2 t}$. Therefore, the solutions with separated variable of the BVP are:

$$u_n(x,t) = C e^{-3\nu_n^2 t} \sin(\nu_n x), \quad \text{where } \nu_n > 0 \text{ solves } \tan(2\nu) = -\nu$$

Exercise 16.

$$\begin{array}{ll} u_t(x,t) = 3u_{xx}(x,t) & 0 < x < 2\,, \quad t > 0 \\ u_x(0,t) = 0 & t > 0 \\ u(2,t) = -u_x(2,t) & t > 0 \end{array}$$

Exercise 17.

$$\begin{array}{ll} u_t(x,t) = 3u_{xx}(x,t) & 0 < x < 2\,, \quad t > 0 \\ u(0,t) = u_x(0,t) & t > 0 \\ u(2,t) = u_x(2,t) & t > 0 \end{array}$$

If $u(x,t) = X(x)T(t)$ solves the BVP, then X and T satisfy the following ODE problems

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X'(0), \quad X(2) = X'(2) \end{cases} \qquad T'(t) + 3\lambda T(t) = 0$$

To find the eigenvalues and eigenfunctions of the X-problem, we consider 3 cases

- *Case* $\lambda < 0$: Set $\lambda = -\nu^2$ with $\nu > 0$. Then $X(x) = C_1 e^{\nu x} + C_2 e^{-\nu x}$. We need $X'(x) = \nu C_1 e^{\nu x} - \nu C_2 e^{-\nu x}$. Then end point conditions $X(0) = X'(0)$ and $X(2) = X'(2)$

lead to the linear system for $C_1,\ C_2$

$$\begin{array}{l} C_1 + C_2 = \nu C_1 - \nu C_2 \\ C_1 e^{2\nu} + C_2 e^{-2\nu} = \nu C_1 e^{2\nu} - \nu C_2 e^{-2\nu} \end{array}$$

In order for this system to have a nontrivial solution ν must satisfy $(1-\nu^2)e^{2\nu} = (1-\nu^2)e^{-2\nu}$. This equation has a solution only when $\nu = 1$. In this case C_1 is arbitrary and $C_2 = 0$. Therefore $\lambda_0 = -1$ is an eigenvalue and $X(x) = e^x$ is an eigenfunction.

- *Case* $\lambda = 0$: In this case the only solution of the X-problem is trivial.
- *Case* $\lambda > 0$: Set $\lambda = \nu^2$ with $\nu > 0$. The general solution of the ODE is $X(x) = C_1 \cos(\nu x) + C_2 \sin(\nu x)$.

  The boundary conditions lead to the system for $C_1,\ C_2$:

$$\begin{array}{l} C_1 = \nu C_2 \\ C_1 \cos(2\nu) + C_2 \sin(2\nu) = -\nu C_1 \sin(2\nu) + \nu C_2 \cos(2\nu) \end{array}$$

  In order for this system to have a nontrivial solution ν must satisfy $(1+\nu^2)\sin(2\nu) = 0$. Thus $2\nu = n\pi$ with $n \in \mathbb{Z}^+$. The eigenvalues are $\lambda_n = \left(\frac{n\pi}{2}\right)^2$ and the eigenfunction

$$X_n(x) = \frac{n\pi}{2} \cos\frac{n\pi x}{2} + \sin\frac{n\pi x}{2}$$

For the eigenvalue $\lambda_0 = -1$, the solution of the T-problem is generated by e^{3t}. For each eigenvalue $\lambda_n = \left(\frac{n\pi}{2}\right)^2$, the solutions of the corresponding T-problem are generated by $T_n(t) = e^{-3\nu_n^2 \pi^2 t/4}$. Therefore, the solutions with separated variable of the BVP are:

$$u_0(x,t) = e^{3t} e^x; \quad u_n(x,t) = e^{-3n^2\pi^2 t/4} \left(\frac{n\pi}{2} \cos\frac{n\pi x}{2} + \sin\frac{n\pi x}{2}\right)$$