FOURIER SERIES PART I: DEFINITIONS AND EXAMPLES

1. Exercises

In each exercise, find the fourier series of the 2π-periodic function f that is given by

Exercise 1. $f(x) = x$ for $-\pi < x < \pi$.

First recall the following formulas that can deduced from integration by parts.

$$\int x\cos(kx)\,dx = \frac{x\sin(kx)}{k} + \frac{\cos(kx)}{k^2} + C$$

$$\int x\sin(kx)\,dx = -\frac{x\cos(kx)}{k} + \frac{\sin(kx)}{k^2} + C$$

The given function f is an odd function, therefore its Fourier coefficients are

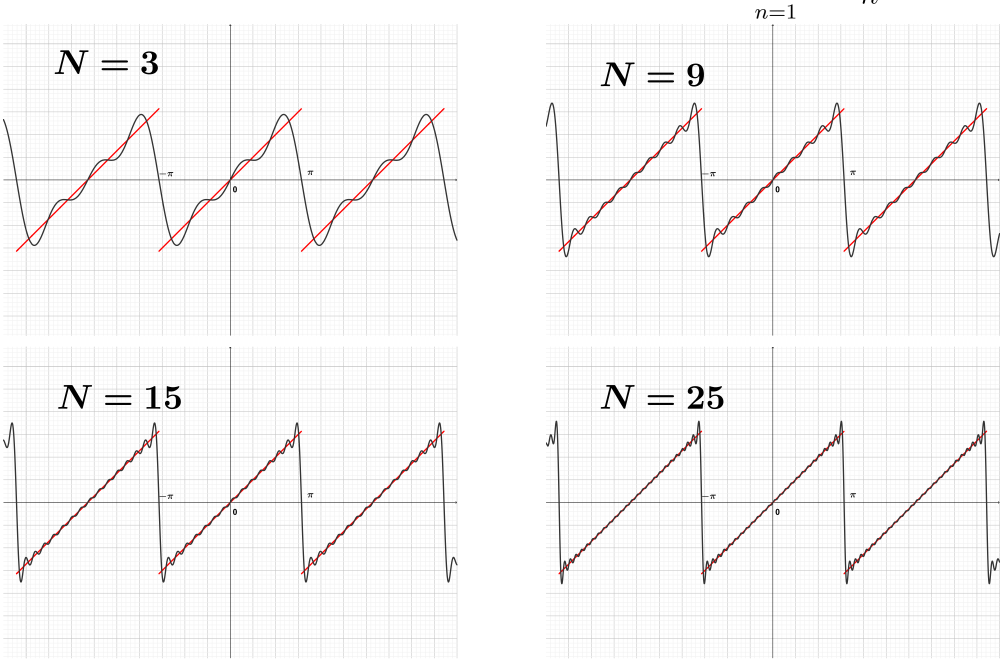
$$a_k = 0 \quad k = 0,\ 1,\ 2,\ 3,\ \cdots$$

$$b_k = \frac{2}{\pi}\int_0^{\pi} x\sin(kx)\,dx = \left[-\frac{x\cos(kx)}{k} + \frac{\sin(kx)}{k^2}\right]_{x=0}^{x=k} = \frac{2(-1)^{k+1}}{k} \quad k = 1,\ 2,\ 3,\ \cdots$$

The Foriers series of f is:

$$\sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k}\sin(kx) = 2\sin x - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \frac{2}{5}\sin(5x) + \cdots$$

Figure 1. Functions f and partial sums of Fourier series $\displaystyle\sum_{n=1}^{N} \frac{2(-1)^{n+1}}{n}\sin(nx)$



Date: February, 2021.

Exercise 2. $f(x) = \pi - x$ for $0 < x < 2\pi$.

Exercise 3. $f(x) = \cos^2 x$

Note that since f is an even function then $b_k = 0$ for all k. The double angle trigonometric formula $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$ shows that the Fourier series of f is $\frac{1}{2} + \frac{1}{2}\cos(2x)$. This means that $a_0 = 1,\ \ a_2 = 1/2$, and $a_k = 0$ for $k \neq 0,\ 2$. If you didn't see this go back to the definition of a_k and compute the integrals.

Exercise 4. $f(x) = \begin{cases} -x & \text{if } -\pi < x < 0 \\ 0 & \text{if } 0 < x < \pi \end{cases}$

Exercise 5. $f(x) = |\cos x|$

The function f is even and therefore the Fourier coefficients $b_n = 0$ for $n = 1,\ 2,\ 3,\ \cdots;$

$$a_0 = \frac{2}{\pi}\int_0^\pi |\cos x|\, dx = \frac{4}{\pi};$$

$$a_1 = \frac{2}{\pi}\int_0^\pi |\cos x| \cos x\, dx = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x\, dx - \int_{\pi/2}^{\pi} \cos^2 x\, dx\right] = 0$$

To compute the integrals involving the a_n's, we can use the trig identity $2\cos(ax)\cos(bx) = \cos(a+b)x\ \cos(a-b)x$. For $n > 1$ we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi |\cos x| \cos(nx)\, dx = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos(nx)\, dx - \int_{\pi/2}^{\pi} \cos x \cos(nx)\, dx\right] \\ &= \frac{1}{\pi}\left[\int_0^{\pi/2} (\cos(n+1)x + \cos(n-1)x)\ dx - \int_{\pi/2}^{\pi} (\cos(n+1)x + \cos(n-1)x)\ dx\right] \\ &= \frac{2}{\pi}\left[\frac{1}{n+1}\sin\frac{(n+1)\pi}{2} + \frac{1}{n-1}\sin\frac{(n-1)\pi}{2}\right] \end{aligned}$$

Since $\sin(m\pi) = 0$ for any integer m, then $a_n = 0$ whenever $n = 2j+1$ is an odd integer and for an even integer $n = 2j$, we have

$$a_{2j} = \frac{2}{\pi}\left[\frac{(-1)^j}{2j+1} + \frac{(-1)^{j-1}}{2j-1}\right] = \frac{4(-1)^{j+1}}{4j^2-1}$$

The Fourier series of f is therefore:

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{(-1)^j}{4j^2-1}\cos(2jx)$$

Exercise 6. $f(x) = |\sin x|$

Exercise 7. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ \sin x & \text{if } 0 < x < \pi \end{cases}$

We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x\,\cos x\,dx = \frac{1}{2\pi}\int_0^{\pi} \sin(2x)\,dx = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x\,\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi} (1-\cos(2x))\,dx = \frac{1}{2}$$

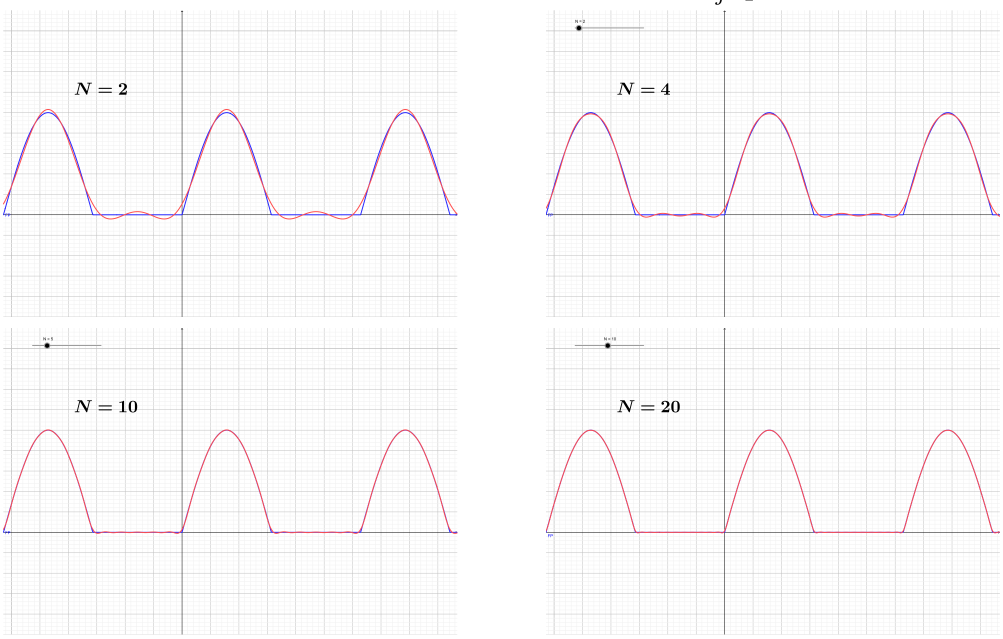
For $n > 1$, we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{\pi} \sin x\,\cos(nx)\,dx = \frac{1}{2\pi}\int_0^{\pi} [\sin((n+1)x) - \sin((n-1)x))\,dx \\ &= \frac{1}{2\pi}\left[\frac{\cos((n-1)x)}{n-1} - \frac{\cos((n+1)x)}{n+1}\right]_{x=0}^{x=\pi} = \frac{(-1)^{n-1}-1}{\pi(n^2-1)} \\ b_n &= \frac{1}{\pi}\int_0^{\pi} \sin x\,\sin(nx)\,dx = \frac{1}{2\pi}\int_0^{\pi} [\cos((n-1)x) - \cos((n+1)x))\,dx \\ &= \frac{1}{2\pi}\left[\frac{\sin((n-1)x)}{n-1} - \frac{\sin((n+1)x)}{n+1}\right]_{x=0}^{x=\pi} = 0 \end{aligned}$$

The Fourier series of f is:

$$\frac{1}{\pi} + \frac{\sin x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}-1}{\pi(n^2-1)}\cos(nx) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{j=1}^{\infty} \frac{1}{4j^2-1}\cos(2jx)$$

FIGURE 2. Partial sums of Fourier series $\dfrac{1}{\pi} + \dfrac{\sin x}{2} - \displaystyle\sum_{j=1}^{N} \frac{2}{\pi(4j^2-1)}\cos(2jx)$



Exercise 8. $f(x) = \begin{cases} 1/(2d) & \text{if } |x| < d \\ 0 & \text{if } d < |x| < \pi \end{cases}$ with d a positive constant.

Exercise 9. $f(x) = \mathrm{e}^{dx}$ for $-\pi < x < \pi$, where d a positive constant

By using integration by parts, we find

$$\begin{aligned} \int \mathrm{e}^{dx}\cos(nx)\,dx &= \frac{d}{d^2+n^2}\mathrm{e}^{dx}\cos(nx) + \frac{n}{d^2+n^2}\mathrm{e}^{dx}\sin(nx) + C \\ \int \mathrm{e}^{dx}\sin(nx)\,dx &= \frac{d}{d^2+n^2}\mathrm{e}^{dx}\sin(nx) - \frac{n}{d^2+n^2}\mathrm{e}^{dx}\cos(nx) + C \end{aligned}$$

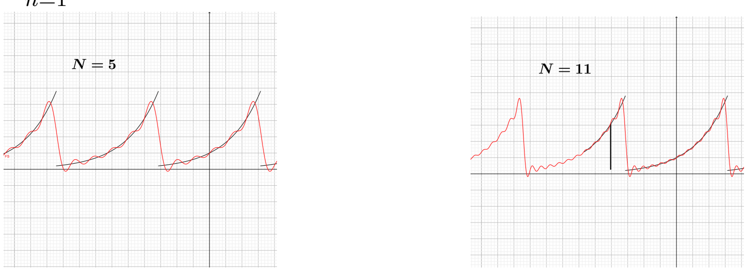
Since $\sin(n\pi) = 0$, $\cos(n\pi) = (-1)^n$, then

$$\begin{aligned} a_0 &= \frac{1}{\pi}\left[\frac{\mathrm{e}^{dx}}{d}\right]_{-\pi}^{\pi} = \frac{2}{\pi d}\frac{\mathrm{e}^{\pi d} - \mathrm{e}^{-\pi d}}{2} = \frac{2}{\pi d}\sinh(\pi d) \\ a_n &= \frac{}{\pi}\frac{d}{d^2+n^2}\left[\mathrm{e}^{dx}\cos(nx)\right]_{-\pi}^{\pi} = \frac{2(-1)^n d}{\pi(d^2+n^2)}\sinh(\pi d) \\ b_n &= \frac{1}{\pi}\frac{-n}{d^2+n^2}\left[\mathrm{e}^{dx}\cos(nx)\right]_{-\pi}^{\pi} = \frac{2(-1)^{n+1} n}{\pi(d^2+n^2)}\sinh(\pi d) \end{aligned}$$

The Fourier series is:

$$\frac{\sinh(\pi d)}{\pi d} + \frac{2\sinh(\pi d)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n d}{n^2+d^2}\cos(nx) - \frac{(-1)^n n}{n^2+d^2}\sin(nx)$$

FIGURE 3. Partial sums of Fourier series $\frac{\sinh(\pi d)}{\pi d} + \frac{2\sinh(\pi d)}{\pi}\sum_{n=1}^{N}\frac{(-1)^n d}{n^2+d^2}\cos(nx) - \frac{(-1)^n n}{n^2+d^2}\sin(nx)$



Exercise 10. $f(x) = \cosh x$ for $-\pi < x < \pi$.

Exercise 11. $f(x) = \cosh x$ for $0 < x < 2\pi$.

Note that $\cosh' x = \sinh x$ and $\sinh' x = \cosh x$. It follows by integration by parts that

$$\int \cosh x\,\cos(nx)\,dx = \frac{\sinh x\,\cos(nx)}{1+n^2} + \frac{n\,\cosh x\,\sin(nx)}{1+n^2} + C$$

$$\int \cosh x\,\sin(nx)\,dx = \frac{\sinh x\,\sin(nx)}{1+n^2} - \frac{n\,\cosh x\,\cos(nx)}{1+n^2} + C$$

The Fourier coefficients are therefore

$$a_n = \frac{\sinh(2\pi)}{\pi(1+n^2)} \quad \text{and} \quad b_n = \frac{(1-\cosh(2\pi))n}{\pi(1+n^2)}$$

The Fourier series is:

$$\frac{\sinh(2\pi)}{2\pi} + \frac{\sinh(2\pi)}{\pi}\sum_{n=1}^{\infty}\frac{\cos(nx)}{(1+n^2)} + \frac{1-\cosh(2\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n\sin(nx)}{(1+n^2)}$$

Exercise 12. $f(x) = \begin{cases} (a-|x|)/(2d) & \text{if } |x| < d \\ 0 & \text{if } d < |x| < \pi \end{cases}$ with d a positive constant.