FOURIER SERIES PART II: CONVERGENCE

1. Exercises

In the following exercises, a 2π-periodic function f is given on the interval $[-\pi, \pi]$. (a.) Find the Fourier series of f: (b.) Find the intervals where $f(x)$ is equal to its Fourier series: (c.) Determine whether the Fourier series converges uniformly

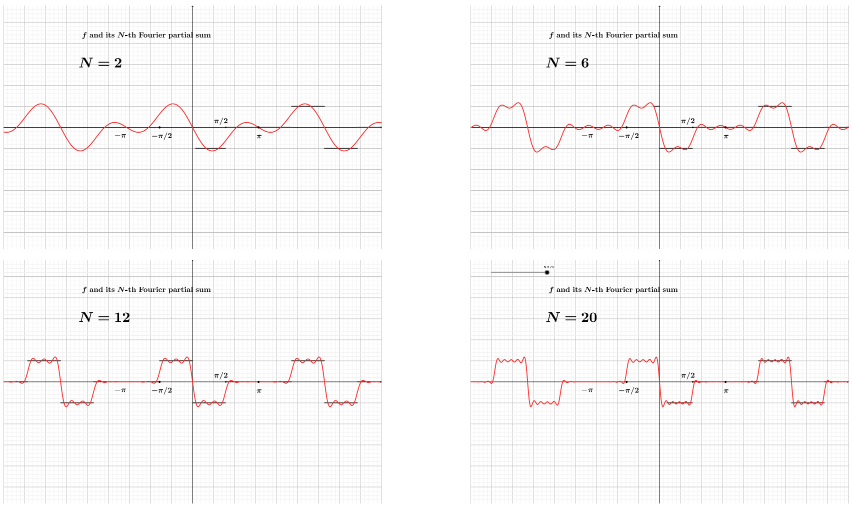
Exercise 1. $f(x) = \begin{cases} -1 & \text{if } 0 < x < \pi/2 \\ 1 & \text{if } -\pi/2 < x < 0 \\ 0 & \text{if } \pi/2 < |x| < \pi \end{cases}$

The function f is odd so $a_n = 0$ for all n and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = -\frac{2}{\pi}\int_0^{\pi/2} \sin(nx)\,dx = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right)$$

Thus the Fourier series of f is

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(n\pi/2) - 1}{n} \sin(nx)$$



Exercise 2. $f(x) = |\sin x|$. Use the Fourier series to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$.

Exercise 3. $f(x) = |\cos x|$. Use the Fourier series to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - 1}$.

The function is even so $b_n = 0$ for all n and

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx = \frac{4}{\pi}\int_0^{\pi/2} \cos x\,dx = \frac{4}{\pi}.$$

Date: February, 2021.

For $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x|\,\cos(nx)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\,\cos(nx)\,dx - \int_{\pi/2}^{\pi}\cos x\,\cos(nx)\,dx\right]$$

Now use the substitution $x = -t + \pi$ to rewrite the second integral as

$$\begin{aligned}\int_{\pi/2}^{\pi}\cos x\,\cos(nx)\,dx &= -\int_{-\pi/2}^{0}\cos(-t+\pi)\,\cos(-nt+n\pi)\,dt \\ &= (-1)^{n+1}\int_0^{\pi/2}\cos t\,\cos(nt)\,dt\end{aligned}$$

It follows (after using the identity $2\cos x\,\cos(nx) = \cos((n+1)x) + \cos((n-1)x))$ that for $n > 1$,

$$\begin{aligned}a_n &= \frac{1+(-1)^n}{\pi}\int_0^{\pi/2} 2\cos x\,\cos(nx)\,dx \\ &= \frac{1+(-1)^n}{\pi}\int_0^{\pi/2} 2\left[\cos((n+1)x) + \cos((n-1)x)\right]\,dx \\ &= \frac{1+(-1)^n}{\pi}\left[\frac{\sin((n+1)\pi/2)}{n+1} + \frac{\sin((n-1)\pi/2)}{n-1}\right]\end{aligned}$$

Since for $n = 2k+1$ (odd), we have $\sin((n+1)\pi/2) = \sin((n-1)\pi/2) = 0$, then $a_{2k+1} = 0$. For $n = 2k$ (even), we have $\sin((2k+1)\pi/2) = (-1)^k$, $\sin((2k-1)\pi/2) = -(-1)^k$. Thus

$$a_{2k} = \frac{2}{\pi}\left[\frac{(-1)^k}{2k+1} - \frac{(-1)^k}{2k-1}\right] = \frac{4}{\pi}\,\frac{(-1)^{k-1}}{4k^2-1}$$

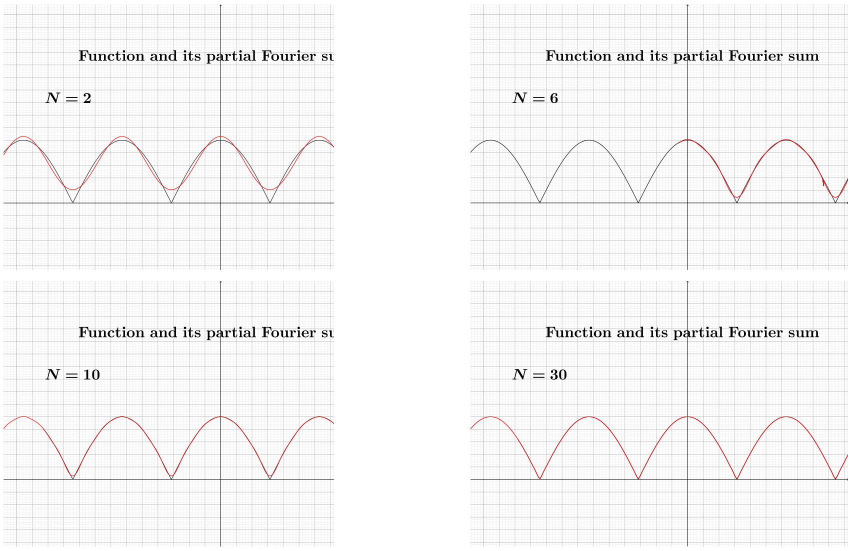
We get the Fourier series representation of f

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{4k^2-1}\cos(2kx)$$

(f is equal to its Fourier series because f is continuous).

For $x = 0$ we have $1 = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{4k^2-1}$. From this we get (after solving for the series)

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{4k^2-1} = \frac{\pi-2}{4}\,.$$



Exercise 4. $f(x) = \cos^2 x$ (thing about a trig. identity)

Exercise 5. $f(x) = \sin^2 x$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

Exercise 6. $f(x) = x^2$. Use the Fourier series to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

We have $b_n = 0$ for all n because the function is even.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3}.$$

For $n \geq 1$, we integration by parts to obtain

$$\int x^2\,dx = \frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3} + C\,.$$

Thus

$$a_n = \frac{2}{\pi}\left[\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right]_0^{\pi} = \frac{4(-1)^n}{n^2}$$

The function is equal to its Fourier series (because it is continuous). In particular

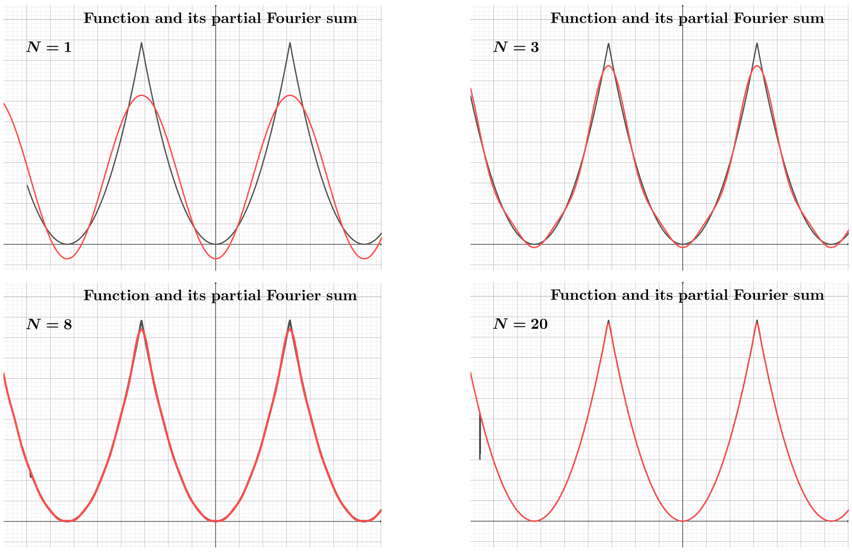
$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx) \qquad \text{for all } x \in [-\pi,\ \pi]$$

Thus for $x = 0$ we get

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \quad\Longrightarrow\quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

For $x = \pi$ we get

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} \quad\Longrightarrow\quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{12}$$



Exercise 7. $f(x) = x(\pi - |x|)$. Use the Fourier series to evaluate $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}$.

Exercise 8. Use the Fourier series for x^2 that you found in exercise 6 to deduce the fourier series of $x^3 - \pi^2 x$ on $[-\pi,\ \pi]$ (use integration of Fourier series).

It follows from the series representation of the function $f(x) = x^2$ (found in exercise 6) that for $x \in [-\pi,\ \pi]$ we have

$$\int_0^x t^2\, dt = \int_0^x \left[\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nt) \right] dt$$

Termwise integration theorem of Fourier series implies

$$\frac{x^3}{3} = \frac{\pi^2 x}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nt)$$

and

$$x^3 - \pi^2 x = 12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nt)$$

Exercise 9. Use the Fourier series you found in exercise 8. To deduce that

$$x^4 - 2\pi^2 x^2 = -\frac{7\pi^4}{15} + 48 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n^4} \quad \text{for } -\pi < x < \pi\ .$$

Deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Exercise 10. Suppose that $f(x)$ has Fourier series $\sum_{n=1}^{\infty} \mathrm{e}^{-n^2} \sin nx$. Find the Fourier series of $f'(x)$ and the Fourier series of $f''(x)$ (justify your answer).

The coefficients of the given trigonometric series are $a_n = 0$ and $b_n = \mathrm{e}^{-n^2}$. Since

$$\sum_{n=1}^{\infty} |a_n| + |b_n| = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2}$$

converge (you can use the ratio test $\sqrt[n]{\mathrm{e}^{-n^2}} = \mathrm{e}^{-n} \longrightarrow 0$ as $n \longrightarrow \infty$). The series converges uniformly and termwise differentiation is valid:

$$f(x) = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2} \sin nx, \quad f'(x) = \sum_{n=1}^{\infty} n\mathrm{e}^{-n^2} \cos nx$$

A similar argument can be applied to find $f''(x) = -\sum_{n=1}^{\infty} n^2 \mathrm{e}^{-n^2} \sin nx$.