FOURIER SERIES PART III: APPLICATIONS

1. Exercises

Exercise 1. (a) Find the Fourier series of the function with period 4 that is defined over $[-2,\ 2]$ by $f(x) = \dfrac{4-x^2}{2}$.

(b) Use Parseval's equality to evaluate the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

(c) Use the integral test to estimate the mean square error E_N when replacing f by its truncated Fourier series $S_N f$.

(d) Find N so that $E_N \le 0.01$ and then find N so that $E_N \le 0.001$

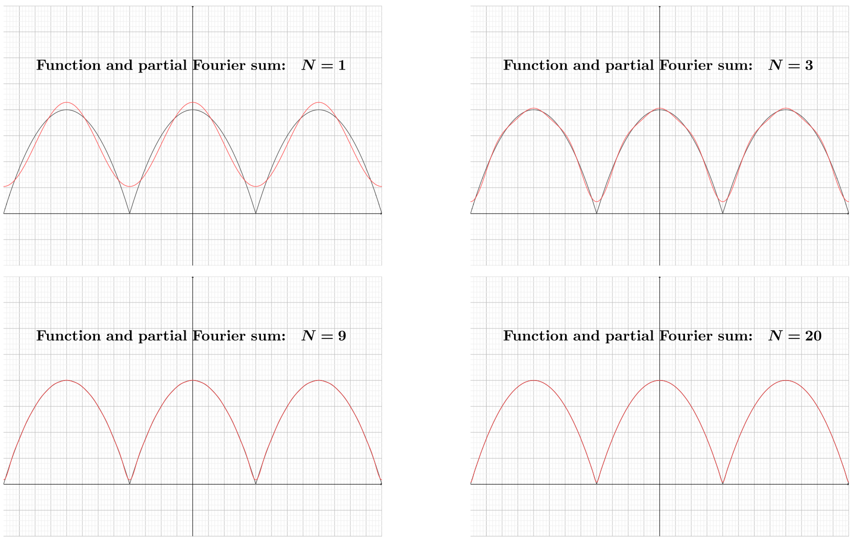
a. The function is even so $b_n = 0$ and $a_n = \dfrac{2}{2}\displaystyle\int_0^2 \frac{4-x^2}{2} \cos\frac{n\pi x}{2}\, dx$.

$$a_0 = \int_0^2 \frac{4-x^2}{2}\, dx = \frac{8}{3}$$

$$a_n = \int_0^2 \frac{4-x^2}{2} \cos\frac{n\pi x}{2}\, dx = \frac{8(-1)^{n+1}}{\pi^2 n^2} \quad \text{for} \quad n \ge 1$$

(the last formula is obtained after integration by parts). Since f is continuous, then

$$\frac{4-x^2}{2} = \frac{4}{3} + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos\frac{n\pi x}{2} \quad \text{for} \quad x \in [-2,\ 2].$$



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b. By using the fact that f is even, $p = 2$, $a_0 = 8/3$ and $a_n = \dfrac{8(-1)^{n+1}}{\pi^2 n^2}$, the Parseval identity

$$\frac{1}{2p}\int_{-p}^{p} f(x)^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

can be rewritten as

$$\frac{1}{2}\int_0^2 \left(\frac{4-x^2}{2}\right)^2 dx = \frac{4^2}{3^2} + \frac{8^2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

so

$$\frac{1}{8}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \frac{16}{9} + \frac{64}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

It follows that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

c. The mean square error E_N is given by

$$E_N^2 = \frac{1}{p}\int_{-p}^{p}\left(f(x) - S_N f(x)\right)^2 dx = \sum_{n=N+1}^{\infty}(a_n^2 + b_n^2)$$

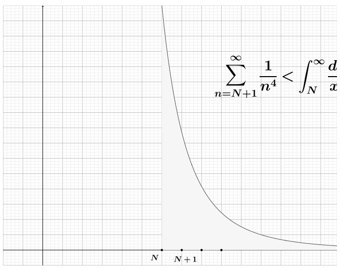
In this case $E_N^2 \le \dfrac{64}{\pi^4}\displaystyle\sum_{n=N+1}^{\infty}\frac{1}{n^4}$. By using the integral test and the function $1/x^4$ (see figure), we find

$$\sum_{n=N+1}^{\infty}\frac{1}{n^4} \le \int_N^{\infty}\frac{1}{x^4}dx = \frac{1}{3N^3}.$$

We have thus the estimate

$$E_N^2 \le \frac{64}{3\pi^4 N^3}$$

d. In order to have $E_N \le 10^{-2}$, it is enough to have



$$\frac{64}{3\pi^4 N^3} \le 10^{-4} \implies N \ge \sqrt[3]{\frac{10^4\, 64}{3\pi^4}} \approx 12.99$$

Therefore taking $N \ge 13$ insures that $S_N f$ approximates f to within 0.01

Exercise 2. (a) Find the Fourier series of the function with period 4 that is defined over $[-2,\ 2]$ by

$$f(x) = \begin{cases} 1 - x & \text{if } 0 \le x \le 2 \\ 1 + x & \text{if } -2 \le x \le 0 \end{cases}$$

(b) Use Parseval's equality to evaluate the series $\displaystyle\sum_{j=0}^{\infty}\frac{1}{(2j+1)^4}$.

(c) Use the integral test to estimate the mean square error E_N when replacing f by its truncated Fourier series $S_N f$.

(d) Find N so that $E_N \le 0.01$ and then find N so that $E_N \le 0.001$

Exercise 3. Find the Fourier sine series of $f(x) = \cos x$ over $[0,\ \pi]$ (What is the Fourier cosine series of $\cos x$ on $[0,\ \pi]$?)

The n-th Fourier sine coefficient of $f(x) = \cos x$ is

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos(x)\,\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} [\sin((n+1)x) + \sin((n-1)x)]\,dx\,.$$

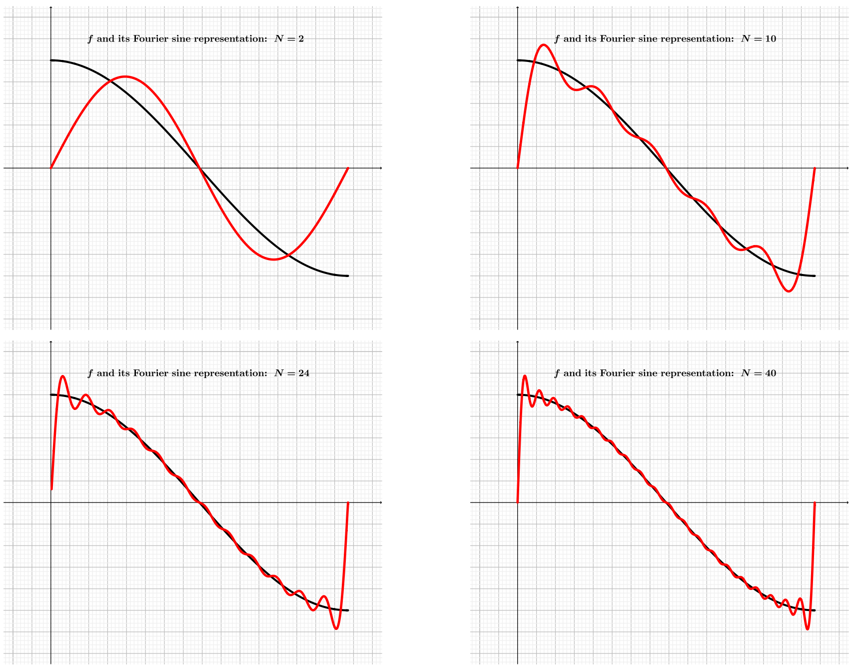
We have $b_1 = \dfrac{-1}{2\pi}\,[\cos(2x)]_0^{\pi} = 0$ and for $n > 1$,

$$b_n = \frac{-1}{\pi}\left[\frac{\cos((n+1)x)}{n+1} - \frac{\cos((n-1)x)}{n-1}\right]_0^{\pi} = \frac{2n(1+(-1)^n)}{\pi(n^2-1)} = \begin{cases} 0 & \text{if } n = 2k+1 \\ \dfrac{8k}{\pi(4k^2-1)} & \text{if } n = 2k \end{cases}$$

The Fourier sine series representation of the function $cosx$ over $[0,\ \pi]$ is

$$\cos x = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{k}{4k^2-1}\,\sin(2kx)\,,\quad \forall x \in (0,\ \pi)\,.$$

The Fourier cosine series of $\cos x$ is $\cos x$ (has only one term).



Exercise 4. Find the Fourier cosine series of $f(x) = \sin x$ over $[0,\ \pi]$ (What is the Fourier sine series of $\sin x$ on $[0,\ \pi]$?)

Exercise 5. Find the Fourier cosine series of $f(x) = x^2$ over $[0,\ 1]$.

We have

$$a_0 = 2\int_0^1 x^2 dx = \frac{2}{3}$$

and for $n \geq 1$ an integration by parts gives

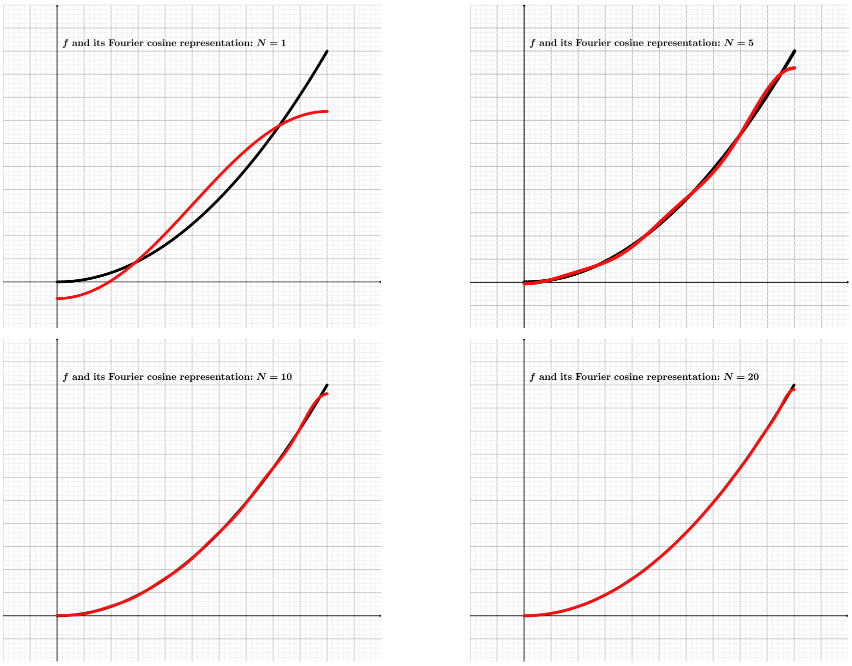
$$\int x^2\cos(n\pi x)\,dx = \frac{x^2\sin(n\pi x)}{n\pi} + \frac{2x\cos(n\pi x)}{n^2\pi^2} - \frac{2\sin(n\pi x)}{n^3\pi^3} + C$$

and

$$a_n = 2\int_0^1 x^2\cos(n\pi x)\,dx = \frac{4(-1)^n}{\pi^2 n^2}$$

The Fourier cosine representation of x^2 over $[0,\ 1]$ is

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)\,.$$



Exercise 6. Find the Fourier sine series of $f(x) = x^2$ over $[0,\ 1]$.

Exercise 7. Find the Fourier cosine series of $f(x) = x\sin x$ over $[0,\ \pi]$.

Exercise 8. Find the Fourier sine series of $f(x) = x\sin x$ over $[0,\ \pi]$.

To find the b_n's, use the identity $2\sin x\,\sin(nx) = \cos((n-1)x) - \cos((n+1)x)$ and then integration by parts. We have

$$b_1 = \frac{1}{\pi}\int_0^{\pi} x(1-\cos(2x))\,dx = \frac{\pi}{2}$$

For $n > 1$,

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\left[\cos((n-1)x) - \cos((n+1)x)\right]\,dx = \frac{4n\left[(-1)^{n+1}-1\right]}{(n^2-1)^2}\,.$$

So $b_{2k+1} = 0$ and $b_{2k} = \dfrac{-16k}{\pi(4k^2-1)^2}$ and

$$x\sin x = \frac{\pi}{2}\sin x - \frac{16}{\pi}\sum_{k=1}^{\infty}\frac{k}{(4k^2-1)^2}\sin(2kx)\,.$$

Exercise 9. Solve the BVP

$$\begin{cases} u_t = u_{xx}, & 0 < x < 2,\quad t > 0 \\ u(0,t) = u(2,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < 2 \end{cases}$$

where

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$$

The solutions with separated variables of the homogeneous part of the BVP are:

$$u_n(x,t) = \mathrm{e}^{-\left(\frac{n\pi}{2}\right)^2 t} \sin \frac{n\pi x}{2}.$$

The series representation of the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \, \mathrm{e}^{-\left(\frac{n\pi}{2}\right)^2 t} \sin \frac{n\pi x}{2}\,.$$

The nonhomogeneous condition $u(x,0) = f(x)$ gives

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{2}.$$

Therefore

$$C_n = \int_0^2 f(x) \sin \frac{n\pi x}{2}\, dx = \int_0^1 \sin \frac{n\pi x}{2}\, dx = \frac{2}{n\pi}\left(1 - \cos \frac{n\pi}{2}\right)$$

The solution of the BVP is:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) \mathrm{e}^{-\left(\frac{n\pi}{2}\right)^2 t} \sin \frac{n\pi x}{2}\,.$$

Exercise 10. Solve the BVP

$$\begin{cases} u_t = u_{xx}, & 0 < x < 2, \quad t > 0 \\ u(0,t) = u(2,t) = 0, & t > 0 \\ u(x,0) = \cos(\pi x), & 0 < x < 2 \end{cases}$$

Exercise 11. Solve the BVP

$$\begin{cases} u_t + u = (0.1) u_{xx}, & 0 < x < \pi, \quad t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0, & t > 0 \\ u(x,0) = \sin x, & 0 < x < \pi \end{cases}$$

If $u(x,t) = X(t)T(t)$ is a nontrivial solution of (HP), then X and T solve the ODE problems

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = 0, \quad X'(\pi) = 0 \end{cases} \qquad T'(t) + (1 + 0.1\lambda) T(t) = 0\,.$$

The eigenvalues and eigenfunctions of the $X-$problem are $\lambda_n = n^2$, $X_n(x) = \cos(nx)$ with $n = 0,\, 1,\, 2,\, \cdots$. The corresponding solutions of the $T-$problem are $T_n(t) = \mathrm{e}^{-(1+0.1\,n^2)t}$. The series representation of the general solution is

$$u(x,t) = C_0 \mathrm{e}^{-t} + \sum_{n=1}^{\infty} C_n \mathrm{e}^{-(1+0.1\,n^2)t} \cos(nx).$$

The nonhomogeneous condition gives

$$u(x,0) = \sin x = C_0 + \sum_{n=1}^{\infty} C_n \cos(nx)$$

(cosine series of $\sin x$ over $[0,\ \pi]$). Hence

$$C_0 = \frac{1}{2}\frac{2}{\pi}\int_0^{\pi} \sin x\, dx = \frac{2}{\pi}, \quad C_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx = \frac{1}{\pi}\int_0^{\pi} \sin(2x)\, dx = 0.$$

For $n \geq 2$, we get

$$C_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos(nx)\, dx = \frac{1}{\pi}\int_0^{\pi} \left[\sin((n+1)x) - \sin((n-1)x)\right] dx = \frac{2[(-1)^{n+1} - 1]}{\pi(n^2 - 1)}$$

Note that $C_{2k+1} = 0$ and $C_{2k} = -\dfrac{4}{\pi(4k^2-1)}$. The solution of the BVP is

$$u(x,t) = \frac{2}{\pi}\mathrm{e}^{-t} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\mathrm{e}^{-(1+0.4\,k^2)t}\cos(2kx)\,.$$

Exercise 12. Consider the BVP modeling heat propagation in a rod where the end points are kept at constant temperatures T_1 and T_2:

$$\begin{cases} u_t = k u_{xx}, & 0<x<L, \quad t>0 \\ u(0,t) = T_1, \;\; u(L,t) = T_2, & t>0 \\ u(x,0) = f(x), & 0<x<L \end{cases}$$

Since T_1 and T_2 are not necessarily zero, we cannot apply directly the method of eigenfunctions expansion. To solve such a problem, we can proceed as follows.

1. Find a function $\alpha(x)$ (independent on time t) so that

$$\alpha''(x) = 0, \quad \alpha(0) = T_1 \;\alpha(L) = T_2\,.$$

2. Let $v(x,t) = u(x,t) - \alpha(x)$. Verify that if $u(x,t)$ solves the given BVP, then $v(x,t)$ solves the following problem

$$\begin{cases} v_t = k v_{xx}, & 0<x<L, \quad t>0 \\ v(0,t) = 0, \;\; v(L,t) = 0, & t>0 \\ v(x,0) = f(x) - \alpha(x), & 0<x<L \end{cases}$$

The v-problem can be solved by the method of separation of variables. The solution u of the original problem is therefore $u(x,t) = v(x,t) + \alpha(x)$.

Exercise 13. Apply the method of described in Exercise 12 to solve the problem

$$\begin{cases} u_t = u_{xx}, & 0<x<2, \quad t>0 \\ u(0,t) = T_1, \;\; u(2,t) = T_2, & t>0 \\ u(x,0) = f(x), & 0<x<2 \end{cases}$$

in the following cases

1. $T_1 = 100$, $T_2 = 0$, $f(x) = 0$.
2. $T_1 = 100$, $T_2 = 100$, $f(x) = 0$.
3. $T_1 = 0$, $T_2 = 100$, $f(x) = 50x$.

Consider the case 2: $T_1 = 100$, $T_2 = 100$, $f(x) = 0$.

The function $\alpha(x)$ satisfies $\alpha''(x) = 0$, $\alpha(0) = 100$, $\alpha(2) = 100$. It follows that $\alpha(x) = 100$. Let $v(x,t) = u(x,t) - \alpha(x)$. The function v satisfies the following BVP

$$\begin{cases} v_t = v_{xx}, & 0<x<2, \quad t>0 \\ v(0,t) = 0, \;\; v(2,t) = 0, & t>0 \\ v(x,0) = -100, & 0<x<2 \end{cases}$$

The solutions with separated variables of the homogeneous part are $\mathrm{e}^{-\frac{\pi^2 n^2}{4}t}\sin\dfrac{n\pi x}{2}$ with $n \in \mathbb{Z}^+$. The series representation of the general solution is

$$v(x,t) = \sum_{n=1}^{\infty} C_n \mathrm{e}^{-\frac{\pi^2 n^2}{4}t}\sin\frac{n\pi x}{2}\,.$$

The nonhomogeneous condition implie

$$v(x,0) = -100 = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{2}\,.$$

Therefore $C_n = \int_0^2 (-100)\sin\frac{n\pi x}{2}\,dx = \frac{200\left[(-1)^n - 1\right]}{n\pi}$. Hence

$$v(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n}\mathrm{e}^{-\frac{\pi^2 n^2}{4}t}\sin\frac{n\pi x}{2} = \frac{-400}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\mathrm{e}^{-\frac{\pi^2(2k+1)^2}{4}t}\sin\frac{(2k+1)\pi x}{2}.$$

The solution $u(x,t) = v(x,t) + \alpha(x)$ of the original BVP is

$$u(x,t) = 100 - \frac{400}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\mathrm{e}^{-\frac{\pi^2(2k+1)^2}{4}t}\sin\frac{(2k+1)\pi x}{2}.$$

In problems 14 to 16, solve the wave propagation problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L, \quad t > 0 \\ u(0,t) = 0, \; u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), \; u_t(x,0) = g(x) & 0 < x < L \end{cases}$$

Exercise 14. $c = 1$, $L = 2$, $f(x) = 0$, $g(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 < x < 2 \end{cases}$

Exercise 15. $c = 1/\pi$, $L = 2$, $f(x) = \sin(\pi x)$, $g(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 < x < 2 \end{cases}$

The BVP in this case is

$$\begin{cases} u_{tt} = \frac{1}{\pi^2}u_{xx}, & 0 < x < 2, \quad t > 0 \\ u(0,t) = 0, \; u(2,t) = 0, & t > 0 \\ u(x,0) = \sin(\pi x), \; u_t(x,0) = g(x) & 0 < x < 2 \end{cases}$$

The solutions with separated variables of the homogeneous part are

$$\cos\frac{nt}{2}\sin\frac{n\pi x}{2}, \quad \sin\frac{nt}{2}\sin\frac{n\pi x}{2}, \quad n \in \mathbb{Z}^+$$

The series representation of the general solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\frac{nt}{2} + B_n\sin\frac{nt}{2}\right]\sin\frac{n\pi x}{2}$$

We have

$$u_t(x,t) = \sum_{n=1}^{\infty}\frac{n}{2}\left[-A_n\sin\frac{nt}{2} + B_n\cos\frac{nt}{2}\right]\sin\frac{n\pi x}{2}$$

The nonhomogeneous conditions imply that

$$u(x,0) = \sin(\pi x) = \sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{2} \quad \text{and} \quad u_t(x,0) = g(x) = \sum_{n=1}^{\infty}\frac{n}{2}B_n\sin\frac{n\pi x}{2}$$

Therefore,

$$A_n = 0 \;\; \text{for} \;\; n \neq 2 \;\; \text{and} \;\; A_2 = 1,$$

and

$$\frac{n}{2}B_n = \int_0^2 g(x)\sin\frac{n\pi x}{2}\,dx = \int_0^1 x\sin\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\,dx.$$

An integration by parts gives

$$\frac{n}{2}B_n = \frac{8\sin(n\pi/2)}{\pi^2 n^2} \quad \Longrightarrow \quad B_n = \frac{16\sin(n\pi/2)}{\pi^2 n^3}$$

The solution of the BVP is

$$u(x,t)=\cos t\,\sin(\pi x)+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^3}\sin\frac{nt}{2}\sin\frac{n\pi x}{2}$$

Exercise 16. $c=2,\ L=\pi,\ f(x)=x\sin x,\ g(x)=\sin(2x)$.

In exercises 17 to 19, solve the wave propagation problem with damping

$$\begin{cases} u_{tt}+2au_t=c^2u_{xx}, & 0<x<L,\quad t>0\\ u(0,t)=0,\ u(L,t)=0, & t>0\\ u(x,0)=f(x),\ u_t(x,0)=g(x) & 0<x<L\end{cases}$$

Exercise 17. $c=1,\ a=.5,\ L=\pi,\ f(x)=0,\ g(x)=x$

The BVP is

$$\begin{cases} u_{tt}+u_t=u_{xx}, & 0<x<\pi,\quad t>0\\ u(0,t)=0,\ u(\pi,t)=0, & t>0\\ u(x,0)=0,\ u_t(x,0)=x & 0<x<\pi\end{cases}$$

If $u(x,t)=X(x)T(t)$ is a nontrivial solution of the homogeneous part, then $X(x)$ and $T(t)$ solve the ODE problems

$$\begin{cases} X''(x)+\lambda X(x)=0\\ X(0)=0,\quad X(\pi)=0\end{cases},\qquad \begin{cases} T''(t)+T'(t)+\lambda T(t)=0\\ T(0)=0\end{cases},$$

where λ is the separation constant. The eigenvalues and eigenfunctions of the SL problem ($X-$problem) are

$$\lambda_n=n^2\,,\qquad X_n=\sin(nx)\qquad n\in\mathbb{Z}^+.$$

For each eigenvalue λ_n, thew corresponding $T-$problem has characteristic equation

$$m^2+m+n^2=0\implies m=\frac{-1}{2}\pm i\,\omega_n\ \text{ with }\omega_n=\frac{\sqrt{4n^2-1}}{2}$$

The $T-$problem has one independent solution given by $T_n(t)=\mathrm{e}^{-t/2}\sin(\omega_n t)$. The solutions with separated variables of the homogeneous part are $\mathrm{e}^{-t/2}\sin(\omega_n t)\sin(nx)$ and a series representation of the general solution is

$$u(x,t)=\sum_{n=1}^{\infty}C_n\mathrm{e}^{-t/2}\sin(\omega_n t)\sin(nx)\,.$$

We have

$$u_t(x,t)=\sum_{n=1}^{\infty}\mathrm{e}^{-t/2}\left[\frac{-C_n}{2}\sin(\omega_n t)+\omega_n C_n\cos(\omega_n t)\right]\sin(nx).$$

So

$$u_t(x,0)=x=\sum_{n=1}^{\infty}\omega_n C_n\sin(nx)\implies \omega_n C_n=\frac{2}{\pi}\int_0^{\pi}x\sin(nx)\,dx=\frac{2(-1)^{n+1}}{n}\,.$$

Thus $C_n=\dfrac{2(-1)^{n+1}}{n\,\omega_n}$ and the solution of the BVP is

$$u(x,t)=\mathrm{e}^{-t/2}\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\,\omega_n}\sin(\omega_n t)\sin(nx)\,.$$

Exercise 18. $c=4,\ a=\pi,\ L=1,\ f(x)=x(1-x),\ g(x)=0$.

Exercise 19. $c=1,\ a=\pi/6,\ L=2,\ f(x)=x\sin(\pi x),\ g(x)=1$.

The BVP is

$$\begin{cases} u_{tt} + \dfrac{\pi}{3} u_t = u_{xx}, & 0 < x < 2, \quad t > 0 \\ \\ u(0,t) = 0, \;\; u(2,t) = 0, & t > 0 \\ u(x,0) = x\sin(\pi x), \;\; u_t(x,0) = 1 & 0 < x < 2 \end{cases}$$

If $u(x,t) = X(x)T(t)$ is a nontrivial solution of the homogeneous part, then $X(x)$ and $T(t)$ solve the ODE problems

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, \quad X(2) = 0 \end{cases}, \qquad T''(t) + \frac{\pi}{3} T'(t) + \lambda T(t) = 0\,,$$

where λ is the separation constant. The eigenvalues and eigenfunctions of the SL problem ($X-$problem) are

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2, \qquad X_n = \sin\frac{n\pi x}{2} \qquad n \in \mathbb{Z}^+.$$

For each eigenvalue λ_n, thew corresponding $T-$problem has characteristic equation

$$m^2 + \frac{\pi}{3} m + \left(\frac{n\pi}{2}\right)^2 = 0 \implies m = \frac{-\pi}{6} \pm i\,\omega_n \;\text{ with }\; \omega_n = \frac{\sqrt{9n^2-1}\,\pi}{3}$$

The $T-$problem has two independent solutions given by $T_{n,1}(t) = \mathrm{e}^{-\pi t/6}\cos(\omega_n t)$ and $T_{n,2}(t) = \mathrm{e}^{-\pi t/6}\sin(\omega_n t)$. The solutions with separated variables of the homogeneous part are $T_{n,1}(t)X_n(x)$ and $T_{n,2}(t)X_n(x)$. The series representation of the general solution is

$$u(x,t) = \mathrm{e}^{-\pi t/6} \sum_{n=1}^{\infty} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right] \sin\frac{n\pi x}{2}\,.$$

We have

$$u_t(x,t) = \mathrm{e}^{-\pi t/6} \sum_{n=1}^{\infty} \left[\left(\omega_n B_n - \frac{\pi}{6} A_n\right)\cos(\omega_n t) - \left(\omega_n A_n + \frac{\pi}{6} B_n\right)\sin(\omega_n t)\right] \sin\frac{n\pi x}{2}\,.$$

The initial conditions are

$$u(x,0) = x\sin(\pi x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{2}, \qquad u_t(x,0) = 1 = \sum_{n=1}^{\infty} \left(\omega_n B_n - \frac{\pi}{6} A_n\right) \sin\frac{n\pi x}{2}\,.$$

Thus

$$A_n = \int_0^2 x\sin(\pi x)\,\sin\frac{n\pi x}{2}\,dx \quad \text{and} \quad \omega_n B_n - \frac{\pi}{6} A_n = \int_0^2 \sin\frac{n\pi x}{2}\,dx\,.$$

Using integration by parts we find

$$A_1 = \frac{-32}{9\pi^2}, \quad A_2 = 1\,, \quad \text{and} \quad A_n = \frac{16}{\pi^2}\frac{\left[(-1)^n - 1\right]n}{(n^2-4)^2} \quad \text{for} \quad n \geq 3.$$

Then

$$B_1 = \frac{92}{27\pi\omega_1}, \quad B_2 = \frac{\pi}{6\omega_2}, \quad \text{and} \quad B_n = \frac{\left[(-1)^n - 1\right]\left(8n^2 - 6(n^2-4)^2\right)}{\pi\omega_n n (n^2-4)^2} \quad \text{for} \quad n \geq 3.$$

The solution of the BVP is

$$u(x,t) = \mathrm{e}^{-t/2}\left[\frac{-32}{9\pi^2}\cos(\omega_1 t) + \frac{92}{27\pi\omega_1}\sin(\omega_1 t)\right]\sin\frac{\pi x}{2} + \mathrm{e}^{-t/2}\left[\cos(\omega_2 t) + \frac{\pi}{6\omega_2}\sin(\omega_2 t)\right]\sin \pi x +$$

$$+ \mathrm{e}^{-t/2} \sum_{n=2}^{\infty} \left[\frac{16}{\pi^2}\cos(\omega_n t) + \frac{8n^2 - 6(n^2-4)^2}{\pi n\,\omega_n}\sin(\omega_n t)\right] \frac{(-1)^n - 1}{(n^2-4)^2}\sin\frac{n\pi x}{2}.$$

In exercises 20 to 22, solve the Laplace equation $\Delta u(x,y)=0$ inside the rectangle $0<x<L$, $0<y<H$ subject the the given boundary conditions.

Exercise 20. $L=H=\pi$, $u(x,0)=x(\pi-x)$, $u(x,\pi)=0$, $u(0,y)=u(\pi,y)=0$.

Exercise 21. $L=\pi$, $H=2\pi$, $u(x,0)=0$, $u(x,2\pi)=x$, $u_x(0,y)=\sin y$, $u_x(\pi,y)=0$.

The BVP is

$$\begin{cases} u_{xx}+u_{yy}, & 0<x<\pi,\quad 0<y<2\pi \\ u(x,0)=0,\; u(x,2\pi)=x, & 0<y<2\pi \\ u_x(0,y)=\sin y,\; u_x(\pi,y)=0 & 0<y<2\pi \end{cases}$$

Seek solution u as $u=v+w$, where $v(x,y)$ and $w(x,y)$ solve the following BVP, respectively.

$$(*)\quad \begin{cases} v_{xx}+v_{yy}, \\ v(x,0)=0,\; v(x,2\pi)=x, \\ v_x(0,y)=0,\; v_x(\pi,y)=0 \end{cases} \quad\text{and}\quad (**)\begin{cases} w_{xx}+w_{yy}, \\ w(x,0)=0,\; w(x,2\pi)=0, \\ w_x(0,y)=\sin y,\; w_x(\pi,y)=0 \end{cases}$$

• v-problem: If $v(x,t)=X(x)Y(y)$ is a nontrivial solution the homogeneous part of $(*)$, then X and Y solve the ODE problems

$$\begin{cases} X''+\lambda X=0 \\ X'(0)=0,\quad X'(\pi)=0 \end{cases} \quad\text{and}\quad \begin{cases} Y''-\lambda Y=0 \\ Y(0)=0 \end{cases}.$$

The eigenvalues and eigenfunctions of the X-problem are

$$\lambda_n=n^2,\quad X_n(x)=\cos(nx),\qquad n=0,\,1,\,2,\,\cdots$$

The corresponding solutions of the Y-problem are

$$Y_0(y)=y \;\text{ for }\; n=0,\quad\text{and}\quad Y_n(y)=\sinh(ny) \;\text{ for }\; n=1,\,2,\,3\cdots$$

The general series solution of BVP $(*)$ is

$$v(x,y)=C_0y+\sum_{n=1}^{\infty}C_n\sinh(ny)\cos(nx)\,.$$

The nonhomogeneous condition gives

$$v(x,2\pi)=x=2\pi C_0+\sum_{n=1}^{\infty}C_n\sinh(2\pi n)\cos(nx)\,.$$

Hence

$$4\pi C_0=\frac{2}{\pi}\int_0^{\pi}x\,dx=\pi \quad\Longrightarrow\quad C_0=\frac{1}{4}$$

and for $n\geq 1$

$$C_n\sinh(2\pi n)=\frac{2}{\pi}\int_0^{\pi}x\cos(nx)\,dx \quad\Longrightarrow\quad C_n=\frac{2\,[(-1)^n-1]}{\pi\,n^2\,\sinh(2\pi n)}$$

The solution of BVP $(*)$ is

$$v(x,y)=\frac{y}{4}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\,\frac{\sinh(ny)}{\sinh(2\pi n)}\,\cos(nx)$$

• w-problem: If $w(x,t)=X(x)Y(y)$ is a nontrivial solution the homogeneous part of $(**)$, then X and Y solve the ODE problems

$$\begin{cases} Y''+\lambda Y=0 \\ Y(0)=0,\quad Y(2\pi)=0 \end{cases} \quad\text{and}\quad \begin{cases} X''-\lambda X=0 \\ X'(\pi)=0 \end{cases}.$$

The eigenvalues and eigenfunctions of the Y-problem are

$$\lambda_n=\left(\frac{n}{2}\right)^2,\quad Y_n(y)=\sin\frac{ny}{2},\qquad n=1,\,2,\,3,\,\cdots$$

The corresponding solutions of the X-problem are

$$X_n(y) = \cosh\frac{n(x-\pi)}{2} \quad \text{for} \quad n = 1,\, 2,\, 3 \cdots$$

The general series solution of BVP $(*)$ is

$$w(x,y) = \sum_{n=1}^{\infty} C_n \cosh\frac{n(x-\pi)}{2}\,\sin\frac{ny}{2}.$$

We have

$$w_x(x,y) = \sum_{n=1}^{\infty} \frac{nC_n}{2} \sinh\frac{n(x-\pi)}{2}\,\sin\frac{ny}{2}.$$

The nonhomogeneous condition gives

$$w_x(0,y) = \sin y = \sum_{n=1}^{\infty} \frac{-nC_n}{2} \sinh\frac{n\pi}{2}\,\sin\frac{ny}{2}\,.$$

Hence

$$C_n = 0 \quad \text{if} \quad n \neq 2 \quad \text{and} \quad -C_2 \sinh\pi = 1\,.$$

The solution of BVP $(**)$ is

$$w(x,y) = -\frac{\cosh(x-\pi)}{\sinh\pi}\,\sin y\,.$$

- The solution u of the original BVP is

$$u(x,y) = w(x,y) + v(x,y) = -\frac{\cosh(x-\pi)}{\sinh\pi}\,\sin y + \frac{y}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\,\frac{\sinh(ny)}{\sinh(2\pi n)}\,\cos(nx)$$

Exercise 22. $L = H = 1$, $u(x,0) = u(x,1) = 0$, $u(0,y) = 1$, $u(1,y) = \sin y$.

Exercise 23. Solve the Laplace equation $\Delta u(r,\theta) = 0$ inside the semicircle of radius 2 ($0 < r < 2$, $0 < \theta < \pi$) subject to the boundary conditions

$$u(r,0) = u(r,\pi) = 0 \quad (0 < r < 2) \quad \text{and} \quad u(2,\theta) = \theta(\pi - \theta) \quad (0 < \theta < \pi)$$

If $u(r,\theta) = R(r)\Theta(\theta)$ is a nontrivial solution of the homogeneous part, then R and Θ solve the ODE problems:

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \\ \Theta(0) = 0, \quad \Theta(\pi) = 0 \end{cases} \quad \text{and} \quad \begin{cases} r^2R''(r) + rR'(r) - \lambda R(r) = 0 \\ R \text{ bounded function} \end{cases}.$$

where λ is the separation constant. The eigenvalues and eigenfunctions of the Θ-problem are:

$$\lambda_n = n^2, \quad \Theta_n(\theta) = \sin(n\theta), \quad n \in \mathbb{Z}^+.$$

The corresponding R-ODE is a Cauchy-Euler with independent solutions r^n and r^{-n}. Since we are interested only in bounded solutions, the solution r^{-n} is not bounded as $r \longrightarrow 0^+$ and will be ignored. The solutions with separation of variables are

$$u_n(r,\theta) = r^n \sin(n\theta) \qquad n \in \mathbb{Z}^+.$$

The series representation of the general solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n r^n \sin(n\theta)\,.$$

Use the nonhomogeneous condition to find the constants C_n.

$$u(2,\theta) = \theta\,(\pi-\theta) = \sum_{n=1}^{\infty} 2^n C_n \sin(n\theta) \quad \Longrightarrow \quad 2^n C_n = \frac{2}{\pi}\int_0^{\pi} \theta\,(\pi-\theta)\sin(n\theta)\,d\theta.$$

An integration by parts gives $2^nC_n = \dfrac{4(1-(-1)^n)}{\pi n^3}$. Therefore the solution of the BVP is

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3} \frac{r^n}{2^n} \sin(n\theta) = \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{r^{2k+1}}{2^{2k+1}(2k+1)^3} \sin((2k+1)\theta)\,.$$

Exercise 24. Solve the Laplace equation $\Delta u(r,\theta) = 0$ inside the semicircle of radius 2 ($0 < r < 2$, $0 < \theta < \pi$) subject to the boundary conditions

$$u_\theta(r,0) = u_\theta(r,\pi) = 0 \quad (0<r<2) \quad \text{and} \quad u(2,\theta) = \theta(\pi-\theta) \quad (0<\theta<\pi)$$

Exercise 25. Solve the Laplace equation $\Delta u(r,\theta) = 0$ inside the quarter of a circle of radius 2 ($0 < r < 2$, $0 < \theta < \pi/2$) subject to the boundary conditions

$$u(r,0) = u(r,\pi/2) = 0 \quad (0<r<2) \quad \text{and} \quad u(2,\theta) = \theta \quad (0<\theta<\pi/2)$$

If $u(r,\theta) = R(r)\Theta(\theta)$ is a nontrivial solution of the homogeneous part, then R and Θ solve the ODE problems:

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \\ \Theta(0) = 0, \quad \Theta(\pi/2) = 0 \end{cases} \quad \text{and} \quad \begin{cases} r^2R''(r) + rR'(r) - \lambda R(r) = 0 \\ R \text{ bounded function} \end{cases}.$$

where λ is the separation constant. The eigenvalues and eigenfunctions of the Θ-problem are:

$$\lambda_n = (2n)^2, \quad \Theta_n(\theta) = \sin(2n\theta), \quad n \in \mathbb{Z}^+.$$

The corresponding R-ODE is a Cauchy-Euler with independent solutions r^{2n} and r^{-2n}. Since we are interested only in bounded solutions, the solution r^{-2n} is not bounded as $r \longrightarrow 0^+$ and will be ignored. The solutions with separation of variables are

$$u_n(r,\theta) = r^{2n}\sin(2n\theta) \qquad n \in \mathbb{Z}^+.$$

The series representation of the general solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n r^{2n} \sin(2n\theta)\,.$$

Use the nonhomogeneous condition to find the constants C_n.

$$u(2,\theta) = \theta = \sum_{n=1}^{\infty} 2^{2n}C_n \sin(2n\theta) \quad \Longrightarrow \quad 2^{2n}C_n = \frac{4}{\pi}\int_0^{\pi/2} \theta\sin(2n\theta)\,d\theta.$$

An integration by parts gives $2^{2n}C_n = \dfrac{(-1)^{n+1}}{n}$. Therefore the solution of the BVP is

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{r^{2n}}{2^{2n}} \sin(2n\theta)\,.$$