STURM-LIOUVILLE PROBLEMS: GENERALIZED FOURIER SERIES

1. Exercises

For exercises 1 to 4: (a) find the eigenvalues and eigenfunctions of the Sturm-Liouville problems; (b) find the generalized Fourier series of the functions $f(x) = 1$ and $g(x) = x$.

Exercise 1. $y'' + \lambda y = 0, \ 0 < x < 1, \ y(0) = 0$ and $y'(1) = 0$

It can be shown that the eigenvalues and eigenfunctions are

$$\lambda_n = \nu_n^2 = \left(\frac{(2n+1)\pi}{2}\right)^2 \quad \text{and} \quad y_n(x) = \sin(\nu_n x) \quad \text{with} \quad n = 0, 1, 2, \cdots$$

We have

$$\|\sin(\nu_n x)\|^2 = \int_0^1 \sin(\nu_n x)^2\, dx = \frac{1}{2}\left[x - \frac{\sin(2\nu_n x}{2\nu_n}\right]_0^1 = \frac{1}{2}$$

The generalized Fourier series of $f(x) = 1$ is $\sum_{n=0}^{\infty} c_n \sin(\nu_n x)$ with

$$c_n = \frac{\langle 1, \sin(\nu_n x)\rangle}{\|\sin(\nu_n x)\|^2} = 2\int_0^1 \sin(\nu_n x)\, dx = \frac{-2}{\nu_n}\left[\cos(\nu_n x)\right]_0^1 = \frac{2}{\nu_n} = \frac{4}{(2n+1)\pi}$$

Therefore, for $x \in (0, \ 1)$

$$1 = \frac{4}{\pi}\sum_0^{\infty} \frac{1}{2n+1} \sin\frac{(2n+1)\pi\, x}{2}.$$

The generalized Fourier series of $g(x) = x$ is $\sum_{n=0}^{\infty} c_n \sin(\nu_n x)$ with

$$\begin{aligned} c_n &= \frac{\langle x, \sin(\nu_n x)\rangle}{\|\sin(\nu_n x)\|^2} = 2\int_0^1 x\sin(\nu_n x)\, dx = 2\left[\frac{-x\cos(\nu_n x)}{\nu_n} + \frac{\sin(\nu_n x)}{\nu_n^2}\right]_0^1 \\ &= \frac{2\sin(\nu_n)}{\nu_n^2} = \frac{8(-1)^n}{(2n+1)^2\pi^2} \end{aligned}$$

Therefore, for $x \in (0, \ 1)$

$$x = \frac{8}{\pi^2}\sum_0^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\frac{(2n+1)\pi\, x}{2}.$$

Exercise 2. $y'' + \lambda y = 0, \ -1 < x < 1, \ y(-1) = y(1)$ and $y'(-1) = y'(1)$ (periodic SL problem)

Exercise 3. $y'' + \lambda y = 0, \ 0 < x < 1, \ y(0) = 0$ and $y(1) + 2y'(1) = 0$

It can be shown that $\lambda \leq 0$ cannot be an eigenvalue of the SL-problem. For $\lambda > 0$, set $\lambda = \nu^2$ with $\nu > 0$, then the general solution of the ODE is $y(x) = A\cos(\nu x) + B\sin(\nu x)$. The condition $y(0) = 0$ implies $A = 0$. The condition $y(1) + 2y'(1) = 0$ leads to $B(\sin\nu + 2\nu\cos\nu) = 0$. In order to have a nontrivial solution y, the parameter ν must satisfy $\sin\nu + 2\nu\cos\nu = 0$ or equivalently $\tan\nu = -2\nu$ (see figure). The eigenvalues and eigenfunctions are

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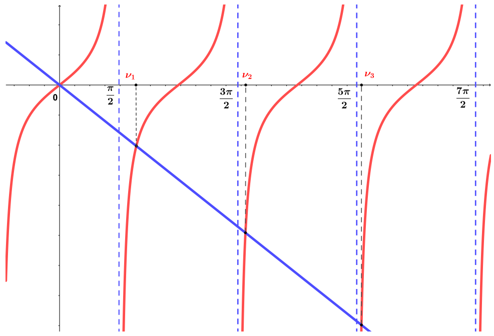

FIGURE 1. *Positive solutions of* $\tan\nu = -2\nu$

$$\lambda_n = \nu_n^2 \quad \text{and} \quad y_n(x) = \sin(\nu_n x) \quad \text{with} \quad \nu_n \text{ is the } n^{\text{th}}\text{root of } \tan\nu = -2\nu$$

The norms of eigenfunctions are

$$\begin{aligned}\|\sin(\nu_n x)\|^2 &= \int_0^1 \sin(\nu_n x)^2\,dx = \frac{1}{2}\left[x - \frac{\sin(2\nu_n x}{2\nu_n}\right]_0^1\\ &= \frac{1}{2}\left(1 - \frac{\sin(2\nu_n)}{2\nu_n}\right) = \frac{1+2\cos^2\nu_n}{2}\end{aligned}$$

The generalized Fourier series of $f(x) = 1$ is $\sum_{n=0}^{\infty} c_n \sin(\nu_n x)$ with

$$c_n = \frac{\langle 1,\, \sin(\nu_n x)\rangle}{\|\sin(\nu_n x)\|^2} = \frac{2}{1+2\cos^2(\nu_n)}\int_0^1 \sin(\nu_n x)\,dx = \frac{2(1-\cos\nu_n)}{\nu_n(1+2\cos^2(2\nu_n)}$$

Therefore, for $x \in (0,\ 1)$

$$1 = \sum_{n=1}^{\infty} \frac{2(1-\cos\nu_n)}{\nu_n(1+2\cos^2(2\nu_n)}\sin(\nu_n x)$$

The generalized Fourier series of $g(x) = x$ is $\sum_{n=0}^{\infty} c_n \sin(\nu_n x)$ with

$$\begin{aligned}c_n &= \frac{\langle x,\, \sin(\nu_n x)\rangle}{\|\sin(\nu_n x)\|^2} = \frac{2}{1+2\cos^2(\nu_n)}\int_0^1 x\sin(\nu_n x)\,dx\\ &= \frac{2}{1+2\cos^2(\nu_n)}\left[\frac{-x\cos(\nu_n x)}{\nu_n} + \frac{\sin(\nu_n x)}{\nu_n^2}\right]_0^1\\ &= \frac{2(-\nu_n\cos\nu_n + \sin\nu_n)}{\nu_n^2(1+2\cos^2(\nu_n))} = \frac{3\sin\nu_n}{\nu_n^2(1+2\cos^2\nu_n)}\end{aligned}$$

Therefore, for $x \in (0,\ 1)$

$$x = \sum_{0}^{\infty} \frac{3\sin\nu_n}{\nu_n^2(1+2\cos^2\nu_n)}\sin(\nu_n x).$$

Exercise 4. $y'' + \lambda y = 0,\ \ 0 < x < 1,\ \ y(0) = y'(0)$ and $y(1) = y'(1)$

Exercise 5. Consider the problem

$$x^2y'' + xy' + \lambda y = 0, \quad 1 < x < L, \quad y(1) = 0, \quad y(L) = 0,$$

with $L > 1$.

(1) Put the ODE in adjoint form: $(py')' + (q + \lambda r)y = 0$ (*Hint*: multiply by $1/x$).
(2) What is the inner product related to this problem?
(3) Find the eigenvalues and eigenfunctions (note: the ODE is Cauchy-Euler).
(4) Find the generalized Fourier series of the function $f(x) = 1$ (*Hint*: when computing the Fourier coefficients c_j, you can use the substitution $t = \ln x$ in the integral).
(5) Same question for the function $g(x) = x$.

(1) Adjoint form of the DE: $(xy')' + \dfrac{\lambda}{x} y = 0$.

(2) The weight associated with the SL-problem is $r(x) = \dfrac{1}{x}$ and the inner product is defined by

$$\langle f, g\rangle_r = \int_1^L f(x)g(x)\frac{1}{x}\,dx.$$

(3) Note that the DE is Cauchy-Euler with characteristic equation $m^2 + \lambda = 0$. Consider 3 cases.

- If $\lambda < 0$, set $\lambda = -\nu^2$ with $\nu > 0$. The general solution of the DE is $y(x) = Ax^\nu + Bx^{-\nu}$. The condition $y(1) = 0$ and $y(L) = 0$ imply $A+B = 0$ and $AL^\nu + BL^{-\nu} = 0$ since $L > 0$, $\nu > 0$, then the only solution is $A = B = 0$ and $\lambda < 0$ cannot be an eigenvalue.
- If $\lambda = 0$. The general solution of the DE is $y(x) = A + B\ln x$. The condition $y(1) = 0$ and $y(L) = 0$ imply $A = 0$ and $B \ln L = 0$ $(B = 0)$. Again $\lambda = 0$ is not an eigenvalue.
- If $\lambda > 0$, set $\lambda = \nu^2$ with $\nu > 0$. The general solution of the DE is $y(x) = A\cos(\nu \ln x) + B\sin(\nu \ln x)$. The condition $y(1) = 0$ gives $A = 0$. Then $y(L) = 0$ implies $B\sin(\nu \ln L) = 0$. To obtain y nontrivial, we need $B \neq 0$ and then $\sin(\nu \ln L) = 0$. Therefore $\nu \ln L = n\pi$ with $n \in \mathbb{Z}^+$.

The eigenvalues and eigenfunctions are:

$$\lambda_n = \nu_n^2 = \left(\frac{n\pi}{\ln L}\right)^2, \quad y_n(x) = \sin(\nu_n \ln x) = \sin\left(n\pi\frac{\ln x}{\ln L}\right), \quad n \in \mathbb{Z}^+$$

(4) The norms of the eigenfunctions are

$$\|y_n\|^2 = \langle y_n, y_n\rangle_r = \int_1^L \sin^2(\nu_n \ln x)\frac{dx}{x} = \int_1^L \sin^2\left(n\pi\frac{\ln x}{\ln L}\right)\frac{dx}{x}.$$

To compute the integral, we use the substitution $t = \ln x$ so that $dt = \dfrac{dx}{x}$ and obtain

$$\|y_n\|^2 = \int_0^{\ln L} \sin^2\left(\frac{n\pi}{\ln L}t\right)\,dt = \frac{\ln L}{2}$$

- Expansion of $f(x) = 1$ in y_n's: We have $1 = \sum_{n=1}^{\infty} c_n y_n(x)$ with

$$\begin{aligned} c_n &= \frac{\langle 1, y_n\rangle_r}{\|y_n\|^2} = \frac{2}{\ln L}\langle 1, y_n\rangle_r = \frac{2}{\ln L}\int_1^L \sin(\nu_n \ln x)\frac{dx}{x} \\ &= \frac{2}{\ln L}\int_0^{\ln L} \sin\left(\frac{n\pi}{\ln L}t\right)\,dt = \frac{2(1-(-1)^n)}{n\pi} \end{aligned}$$

Hence

$$1 = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\,\sin\left(n\pi\,\frac{\ln x}{\ln L}\right)$$

- Expansion of $g(x) = x$ in y_n's: We have $x = \sum_{n=1}^{\infty} c_n y_n(x)$ with

$$\begin{aligned} c_n &= \frac{\langle x, y_n\rangle_r}{\|y_n\|^2} = \frac{2}{\ln L}\langle x, y_n\rangle_r = \frac{2}{\ln L}\int_1^L x\sin(\nu_n \ln x)\frac{dx}{x} \\ &= \frac{2}{\ln L}\int_0^{\ln L} \mathrm{e}^t \sin\left(\frac{n\pi}{\ln L}t\right)\,dt \end{aligned}$$

Since $\int \mathrm{e}^t\sin(at)dt = \frac{\mathrm{e}^t\left[\sin(at) - a\cos(at)\right]}{1+a^2} + C$, then

$$c_n = \frac{n\pi\left[1-(-1)^n L\right]}{\ln^2 L + n^2\pi^2}$$

and

$$x = \sum_{n=1}^{\infty}\frac{n\pi\left[1-(-1)^n L\right]}{\ln^2 L + n^2\pi^2}\,\sin\left(n\pi\,\frac{\ln x}{\ln L}\right)$$

Exercise 6. Same questions as in Exercise 5 for the SL-problem

$$x^2y'' + xy' + \lambda y = 0, \quad 1 < x < L, \quad y'(1) = 0, \quad y'(L) = 0,$$

Exercise 7. Solve the BVP

$$\begin{array}{ll} u_t = 2u_{xx} & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0 & t > 0 \\ 2u(\pi,t) + u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = \sin x & 0 < x < \pi \end{array}$$

We proceed by finding solutions with separated variables $u(x,t) = X(x)T(t)$ of the homogeneous part. This leads to the following ODE problems for X and T, where λ is the separation constant:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, \quad 2X(\pi) + X'(\pi) = 0 \end{cases}, \qquad T'(t) + 2\lambda T(t) = 0\,.$$

It can be verified that $\lambda \le 0$ cannot be an eigenvalue of the X-problem. For $\lambda > 0$, set $\lambda = \nu^2$ with $\nu > 0$. The general solution of the ODE is $X(x) = A\cos(\nu x) + B\sin(\nu x)$. The condition $X(0) = 0$ implies $A = 0$. Then the condition $2X(\pi) + X'(\pi) = 0$ leads to $B(2\sin(\nu\pi) + \nu\cos(\nu\pi)) = 0$. For $B \neq 0$, ν needs to satisfy $2\sin(\nu\pi) + \nu\cos(\nu\pi) = 0$ or $\tan(\nu\pi) = -\frac{\nu}{2}$. This equation has infinitely many solutions, for every $n \in \mathbb{Z}^+$, the equation has a unique solution ν_n in the interval $\left(\frac{2n-1}{2}, \frac{2n+1}{2}\right)$ (see figure). The eigenvalues and eigenfunctions of the X-problem are:

$$\lambda_n = \nu_n^2, \quad \text{with} \quad \nu_n \in \left(\frac{2n-1}{2}, \frac{2n+1}{2}\right) \quad \tan(\nu_n\pi) = -\frac{\nu_n}{2}, \quad \text{and} \quad X_n(x) = \sin(\nu_n x).$$

The corresponding solutions of the T-problem are $T_n(t) = \mathrm{e}^{-2\nu_n^2 t}$. The solutions with separated

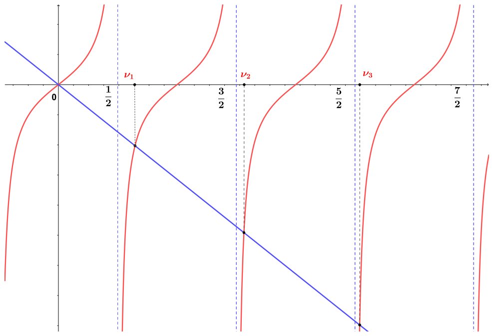

FIGURE 2. *Positive solutions of* $\tan(\nu\pi) = -\nu/2$

variables of the homogeneous part of the BVP are $e^{-2\nu_n^2 t}\sin(\nu_n x)$. The series representation of the general solution of the HP is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-2\nu_n^2 t}\sin(\nu_n x)\,.$$

In order for u to satisfy the completed BVP, we need to have

$$u(x,0) = \sin x = \sum_{n=1}^{\infty} c_n \sin(\nu_n x).$$

The series is the generalized Fourier expansion of $\sin x$ in eigenfunctions of the X-problem. Thus

$$c_n = \frac{\langle \sin x, \sin(\nu_n x)\rangle}{\|\sin(\nu_n x)\|^2}.$$

We have

$$\begin{aligned}\|\sin(\nu_n x)\|^2 &= \int_0^\pi \sin^2(\nu_n x)\,dx = \frac{1}{2}\int_0^\pi (1-\cos(2\nu_n x))\,dx \\ &= \frac{1}{2}\left(\pi - \frac{\sin(2\nu_n\pi)}{2\nu_n}\right) = \frac{2\pi + \cos^2(\nu_n\pi)}{4}.\end{aligned}$$

and

$$\begin{aligned}\langle \sin x, \sin(\nu_n x)\rangle &= \int_0^\pi \sin x \sin(\nu_n x)\,dx = \frac{1}{2}\int_0^\pi \left[\cos(\nu_n - 1)x - \cos(\nu_n+1)x\right]\,dx \\ &= \frac{-\sin(\nu_n\pi)}{2(\nu_n-1)} + \frac{\sin(\nu_n\pi)}{2(\nu_n+1)} = \frac{-\sin(\nu_n\pi)}{\nu_n^2-1}.\end{aligned}$$

Hence $c_n = \dfrac{-4\sin(\nu_n\pi)}{(\nu_n^2-1)(2\pi+\cos^2(\nu_n\pi))}$ and the solution of the BVP is

$$u(x,t) = -4\sum_{n=1}^{\infty}\frac{\sin(\nu_n\pi)}{(\nu_n^2-1)(2\pi+\cos^2(\nu_n\pi))}e^{-2\nu_n^2 t}\sin(\nu_n x)\,.$$

Exercise 8. Solve the BVP

$$\begin{array}{ll} u_t = u_{xx} & 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0 & t > 0 \\ u(\pi,t) = u_x(\pi,t) & t > 0 \\ u(x,0) = 1 & 0 < x < \pi \end{array}$$

Exercise 9. Solve the BVP

$$\begin{array}{ll} u_{tt} = c^2 u_{xx} & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0 & t > 0 \\ u(\pi,t) - u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = \sin x & 0 < x < \pi \\ u_t(x,0) = 0 & 0 < x < \pi \end{array}$$

We proceed by finding solutions with separated variables $u(x,t) = X(x)T(t)$ of the homogeneous part. This leads to the following ODE problems for X and T, where λ is the separation constant:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, \quad X(\pi) = X'(\pi) \end{cases}, \qquad \begin{cases} T''(t) + c^2 \lambda T(t) = 0\,, \\ T'(0) = 0 \end{cases}.$$

To find the eigenvalues of the X-problem, we consider three cases.

- $\lambda < 0$. Set $\lambda = -\mu^2$ with $\mu > 0$. In this case the general solution of the ODE is $X(x) = A\cosh(\mu x) + B\sinh(\mu x)$. The condition $X(0) = 0$ implies $A = 0$, then the second condition leads to $B\sinh(\mu\pi) = B\mu\cosh(\mu\pi)$. For $B \neq 0$, the parameter μ must satisfy $\sinh(\mu\pi) = \mu\cosh(\mu\pi)$ or equivalently $e^{2\mu\pi} = \dfrac{1+\mu}{1-\mu}$. This equation has a unique positive solution μ_0 with $\mu_0 \in (0,\ 1)$. In fact $\mu_0 \approx 0.996$. Hence $\lambda_0 = -\mu_0^2$ is an eigenvalue with corresponding eigenfunction $X_0(x) = \sinh(\mu_0 x)$.
- It can be verified that $\lambda = 0$ is not an eigenvalue.
- $\lambda > 0$. Set $\lambda = \nu^2$ with $\nu > 0$. The general solution of the ODE is $X(x) = A\cos(\nu x) + B\sin(\nu x)$. The condition $X(0) = 0$ implies $A = 0$. Then the condition $X(\pi) = X'(\pi)$ leads to $B\sin(\nu\pi) = B\nu\cos(\nu\pi))$. For $B \neq 0$, ν needs to satisfy $\sin(\nu\pi) = \nu\cos(\nu\pi)$ or $\tan(\nu\pi) = \nu$. This equation has infinitely many solutions, for every $n \in \mathbb{Z}^+$, the equation has a unique solution ν_n in the interval $\left(n,\ \dfrac{2n+1}{2}\right)$. The eigenvalues and eigenfunctions of the X-problem are $\lambda_n = \nu_n^2$ and the corresponding eigenfunction $X_n(x) = \sin(\nu_n x)$.

For the negative eigenvalue $\lambda_0 = -\mu_0^2$, the corresponding T-equation becomes $T'' - c^2\mu_0^2 T = 0$ with general solution $T(t) = A\cosh(c\mu_0 t) + B\sinh(c\mu_0 t)$. The condition $T'(0) = 0$ implies $B = 0$. The solution of HP of the BVP with separated variables is

$$u_0(x,t) = \cosh(c\mu_0 t)\sinh(\mu_0 x).$$

For the positive eigenvalues $\lambda_n = \nu_n^2$, the corresponding T-equation becomes $T'' + c^2\nu_n^2 T = 0$ with general solution $T(t) = A\cos(c\nu_n t) + B\sin(c\nu_n t)$. The condition $T'(0) = 0$ implies $B = 0$. The solution of HP of the BVP with separated variables is

$$u_n(x,t) = \cos(c\nu_n t)\sin(\nu_n x).$$

The series representation of the general solution of HP is therefore

$$u(x,t) = c_0\cosh(c\mu_0 t)\sinh(\mu_0 x) + \sum_{n=1}^{\infty} c_n\cos(c\nu_n t)\sin(\nu_n x).$$

Now we use the nonhomogeneous condition to find the constants c_n's so that u solves the complete BVP.

$$u(x,0) = \sin x = c_0\sinh(\mu_0 x) + \sum_{n=1}^{\infty} c_n\sin(\nu_n x).$$

Therefore

$$c_0 = \frac{\langle \sin x,\ \sinh(\mu_0 x)\rangle}{\|\sinh(\mu_0 x)\|^2} \quad \text{and} \quad c_n = \frac{\langle \sin x,\ \sin(\nu_n x)\rangle}{\|\sin(\nu_n x)\|^2}\,.$$

We have

$$\begin{aligned}
\|\sinh(\mu_0 x)\|^2 &= \int_0^\pi \sinh(\mu_0 x)^2 dx = \frac{1}{2}\int_0^\pi [\cosh(2\mu_0 x) - 1]\ dx \\
&= \frac{\sinh(2\mu_0\pi)}{4\mu_0} - \frac{\pi}{2} = \frac{\cosh^2(\mu_0\pi) - \pi}{2}; \\
\|\sin(\nu_n x)\|^2 &= \int_0^\pi \sin(\nu_n x)^2 dx = \frac{1}{2}\int_0^\pi [1 - \cos(2\nu_n x)]\ dx \\
&= \frac{\pi}{2} - \frac{\sin(2\nu_n\pi)}{4\nu_n} = \frac{\pi - \cos^2(\nu_n\pi)}{2}; \\
\langle \sin x,\, \sinh(\mu_0 x)\rangle &= \int_0^\pi \sin x\ \sinh(\mu_0 x) dx \\
&= \frac{\mu_0^2}{1+\mu_0^2}\left[\frac{\sin x\ \cosh(\mu_0 x)}{\mu_0} - \frac{\cos x\ \sinh(\mu_0 x)}{\mu_0^2}\right]_0^\pi \\
&= \frac{\sinh(\mu_0\pi)}{1+\mu_0^2}; \\
\langle \sin x,\, \sin(\nu_n x)\rangle &= \int_0^\pi \sin x\ \sin(\nu_n x) dx \\
&= \frac{1}{1-\nu_n^2}\left[-\cos x \sin(\nu_n x) + \nu_n \sin x \cos(\nu_n x)\right]_0^\pi \\
&= \frac{\sin(\nu_n\pi)}{1-\nu_n^2}.
\end{aligned}$$

Hence

$$c_0 = \frac{2\sinh(\mu_0\pi)}{(1+\mu_0^2)(\cosh^2(\mu_0\pi) - \pi)} \quad \text{and} \quad c_n = \frac{2\sin(\nu_n\pi)}{(1-\nu_n^2)(\pi - \cos^2(\nu_n\pi))}.$$

The solution of the BVP is:

$$u(x,t) = \frac{2\sinh(\mu_0\pi)\ \cosh(c\mu_0 t)\sinh(\mu_0 x)}{(1+\mu_0^2)(\cosh^2(\mu_0\pi) - \pi)} + \sum_{n=1}^{\infty} \frac{2\sin(\nu_n\pi)\ \cos(c\nu_n t)\sin(\nu_n x)}{(1-\nu_n^2)(\pi - \cos^2(\nu_n\pi))}$$

Exercise 10. Solve the BVP

$$\begin{array}{ll}
u_{tt} = c^2 u_{xx} & 0 < x < \pi, \quad t > 0, \\
u(0,t) = 0 & t > 0 \\
u(\pi,t) - u_x(\pi,t) = 0 & t > 0 \\
u(x,0) = 0 & 0 < x < \pi \\
u_t(x,0) = f(x) & 0 < x < \pi
\end{array}$$

where

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < (\pi/2), \\ 1 & \text{if } (\pi/2) < x < \pi. \end{cases}$$

Exercise 11. Solve the BVP (here a is a positive constant)

$$\begin{array}{ll} u_{tt} = u_{xx} & 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = au(0,t) & t > 0 \\ u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < \pi \\ u_t(x,0) = 1 & 0 < x < \pi \end{array}$$

We proceed by finding solutions with separated variables $u(x,t) = X(x)T(t)$ of the homogeneous part. This leads to the following ODE problems for X and T, where λ is the separation constant:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = aX(0), \quad X'(\pi) = 0 \end{cases}, \qquad \begin{cases} T''(t) + \lambda T(t) = 0\,, \\ T(0) = 0 \end{cases}.$$

To find the eigenvalues of the X-problem, we consider three cases.

- $\lambda < 0$. Set $\lambda = -\mu^2$ with $\mu > 0$. In this case the general solution of the ODE is $X(x) = A\cosh(\mu x) + B\sinh(\mu x)$. We have $X'(x) = \mu A\sinh(\mu x) + \mu B\cosh(\mu x)$. The condition $X'(0) = aX(0)$ implies $B\mu = Aa$, then the second condition $X'(\pi) = 0$ leads to $A(\mu\sinh(\mu\pi) + a\cosh(\mu\pi)) = 0$. Since $\mu > 0$, $a > 0$, then $\mu\sinh(\mu\pi) + a\cosh(\mu\pi) > 0$ and $A = 0$. This implies $X = 0$ and $\lambda < 0$ cannot be an eigenvalue.
- It can be verified that $\lambda = 0$ is not an eigenvalue.
- $\lambda > 0$. Set $\lambda = \nu^2$ with $\nu > 0$. The general solution of the ODE is $X(x) = A\cos(\nu x) + B\sin(\nu x)$. We have $X'(x) = -\nu A\sin(\nu x) + \nu B\cos(\nu x)$ The condition $X(0) = 0$ implies $A = 0$. Then the condition $X'(0) = aX(0)$ implies $aA = \nu B$. Then the condition $X'(\pi) = 0$ leads to $A(a\cos(\nu\pi) - \nu\sin(\nu\pi)) = 0$. For $A \neq 0$, ν needs to satisfy $\nu\sin(\nu\pi) = a\cos(\nu\pi)$ or $\tan(\nu\pi) = \frac{a}{\nu}$. This equation has infinitely many solutions, for every $n \in \mathbb{Z}^+$, the equation has a unique solution ν_n in the interval $\left(n-1, \ \frac{2n-1}{2}\right)$. The eigenvalues and eigenfunctions of the X-problem are $\lambda_n = \nu_n^2$ and the corresponding eigenfunction $X_n(x) = \nu_n\cos(\nu_n x) + a\sin(\nu_n x)$.

For the eigenvalue $\lambda_n = \nu_n^2$, the corresponding T-equation becomes $T'' + \nu_n^2 T = 0$ with general solution $T(t) = A\cos(c\nu_n t) + B\sin(c\nu_n t)$. The condition $T(0) = 0$ implies $A = 0$. The solution of HP of the BVP with separated variables is

$$u_n(x,t) = \sin(\nu_n t)\left[\nu_n\cos(\nu_n x) + a\sin(\nu_n x)\right].$$

The series representation of the general solution of HP is therefore

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\nu_n t)\left[\nu_n\cos(\nu_n x) + a\sin(\nu_n x)\right].$$

Now we use the nonhomogeneous condition to find the constants c_n's so that u solves the complete BVP. We have

$$u_t(x,0) = 1 = \sum_{n=1}^{\infty} \nu_n c_n\left[\nu_n\cos(\nu_n x) + a\sin(\nu_n x)\right].$$

Therefore

$$\nu_n c_n = \frac{\langle 1, \ \nu_n\cos(\nu_n x) + a\sin(\nu_n x)\rangle}{\|\nu_n\cos(\nu_n x) + a\sin(\nu_n x)\|^2}.$$

Exercise 12. Solve the BVP

$$\begin{array}{ll} u_t = (1+x)^2 u_{xx} & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0 \quad u(1,t) = 0 & t > 0 \\ u(x,0) = x(1-x)\sqrt{1+x} & 0 < x < 1 \end{array}$$