Spring 2022 – MAP4401
Test 2 Solutions

Problem 1. Let f be the 2π-periodic function defined over $[-\pi,\ \pi]$ by

$$f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } -\pi < x < 0 \end{cases}$$

Find the following Fourier coefficients of f: a_0, a_1, a_2, and b_1

Using the formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\cos(nx)\,dx \quad \text{and}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\sin(nx)\,dx$$

we find

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}\,, \quad a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x\cos x\,dx = \frac{1}{2\pi}\left[\sin^2 x\right]_0^{\pi} = 0$$

$$a_2 = \frac{1}{\pi}\int_0^{\pi} \sin x\cos(2x)\,dx = \frac{1}{2\pi}\int_0^{\pi} \left(\sin(3x) - \sin x\right)\,dx = \frac{1}{2\pi}\left[\frac{-\cos(3x)}{3} + \cos x\right]_0^{\pi} = \frac{-2}{3\pi}$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x\,dx = \frac{1}{2\pi}\int_0^{\pi} \left(1 - \cos(2x)\right)\,dx = \frac{1}{2\pi}\left[x - \frac{\sin(2x)}{s}\right]_0^{\pi} = \frac{1}{2}$$

Problem 2. Consider the periodic function $f(x)$ with period 2, given on the interval $[-1,\ 1]$ by $f(x) = x^2$. The function f is an even continuous function on $\mathbb{R}$ and with Fourier series:

$$f(x) = \frac{1}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos(n\pi x)\,.$$

(a) Use integration of Fourier series to find the Fourier series of the function $g(x)$ with period 2 and defined on the interval $[-1,\ 1]$ by $g(x) = x^3 - x$.

For $x \in [-1,\ 1]$ we have $\int_0^x f(t)dt = \int_0^x t^2 dt = \frac{x^3}{3}$. If we replace $f(t)$ by its Fourier series we have then

$$\frac{x^3}{3} = \int_0^x \left[\frac{1}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos(n\pi t)\right] dt = \frac{x}{3} - \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}\sin(n\pi x)$$

It follows that

$$x^3 - x = \frac{12}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}\sin(n\pi x)\,.$$

(b) Use the Fourier series representation of f at an appropriate values of x to evaluate the numerical series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{25} + \cdots$$

Since f is continuous at 0, then $f(0) = 0$ is equal to the value of its Fourier series at 0. Thus

$$0 = \frac{1}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

It follows that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

(c) Use the Fourier series representation of g at an appropriate values of x to evaluate the numerical series

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{5^3} - \cdots$$

Since g is continuous at $x = 1/2$, then $g(1/2) = (1/8) - (1/2)$ is equal to the value of its Fourier series at $1/2$. Thus

$$\frac{1}{8} - \frac{1}{2} = \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin \frac{n\pi}{2} = \frac{12}{\pi^3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$$

It follows that $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}$

Problem 3. Given a piecewise smooth function $f(x)$ with period $2L$ and with Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

1. Write Parseval's identity for the function f.

$$\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{2L}\int_{-L}^{L} f(x)^2 dx\,.$$

2. The 2π-periodic function $f(x)$ that is defined on the interval $[-\pi,\ \pi]$ by $f(x) = \pi^2 x - x^3$ has Fourier series

$$12 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin(nx)\,.$$

Use Parseval's identity to evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n^6}$

$$\frac{12^2}{2} \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi^2 x - x^3)^2 dx = \frac{1}{2\pi}\left[\frac{\pi^4 x^3}{3} - \frac{2\pi^2 x^5}{5} + \frac{x^7}{7}\right]_{-\pi}^{\pi} = \frac{8\pi^6}{3\cdot 5\cdot 7}\,.$$

It follows that $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$

Problem 4. (49 pts) Solve the following boundary value problem

$$\begin{cases} u_{tt} + 2u_t = 16u_{xx}, & 0 < x < 2, \quad t > 0 \\ u(0,t) = u(2,t) = 0, & t > 0 \\ u(x,0) = 0, & 0 < x < 2 \\ u_t(x,0) = g(x), & 0 < x < 2 \end{cases}$$

with $g(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{for } 1 < x < 2 \end{cases}$

The homogeneous and nonhomogeneous parts of the BVP are:

$$\text{(HP)}: \quad \begin{cases} u_{tt} + 2u_t = 16u_{xx} \\ u(0,t) = u(2,t) = 0 \\ u(x,0) = 0 \end{cases} \qquad \text{(NHP)}: \quad u_t(x,0) = g(x)$$

If $u(x,t) = X(x)T(t)$ solves (HP), then the functions X and T solve the ODE problems

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, \ X(2) = 0 \end{cases} \quad \text{and} \quad \begin{cases} T''(t) + 2T'(t) + 16\lambda T(t) = 0 \\ T(0) = 0 \end{cases}$$

where λ is the separation constant. The eigenvalues and eigenfunctions of the X-problem are

$$\lambda_n = \nu_n^2 \quad \text{and} \quad X_n(x) = \sin(\nu_n x), \quad \text{with} \quad \nu_n = \frac{n\pi}{2}, \quad n \in \mathbb{Z}^+.$$

For $\lambda = \nu_n^2$, the characteristic equation of the T-problem is $m^2 + 2m + 16\nu_n^2 = 0$ with complex conjugate roots

$$m_{1,2} = -1 \pm i\omega_n \quad \text{with} \quad \omega_n == \sqrt{16\nu_n^2 - 1}\,.$$

The general solution of the T-equation is $T(t) = \mathrm{e}^{-t}(A\cos(\omega_n t) + B\sin(\omega_n t))$. Since in addition $T(0) = 0$, then $A = 0$.

The solutions with separated variables of (HP) are: $\mathrm{e}^{-t}\sin(\omega_n t)\sin(\nu_n x)$. The principle of suerposition gives the general solution as

$$u(x,t) = \sum_{n=1}^{\infty} C_n \mathrm{e}^{-t}\sin(\omega_n t)\sin(\nu_n x)\,.$$

We have

$$u_t(x,t) = \sum_{n=1}^{\infty} C_n \left[-\mathrm{e}^{-t}\sin(\omega_n t) + \omega_n \mathrm{e}^{-t}\cos(\omega_n t)\right]\sin(\nu_n x)\,.$$

Hence

$$u_t(x,0) = \sum_{n=1}^{\infty} C_n\omega_n \sin(\nu_n x) = g(x)\,.$$

Therefore

$$C_n\omega_n = \frac{2}{2}\int_0^2 g(x)\sin\frac{n\pi x}{2}dx = \int_0^1 \sin\frac{n\pi x}{2} = \frac{2(1-\cos(n\pi/2))}{n\pi}$$

Thus $C_n = \dfrac{2(1-\cos(n\pi/2))}{\pi n\omega_n}$ and the solution of the BVP is

$$u(x,t) = \frac{\mathrm{e}^{-t}}{\pi}\sum_{n=1}^{\infty}\frac{2(1-\cos(n\pi/2))}{n\omega_n}\sin(\omega_n t)\sin(\nu_n x)$$