TEST 3 - SOLUTIONS

Exercises from LN 9

Exercise 2. $y'' + \lambda y = 0,\ -1 < x < 1,\ y(-1) = y(1)$ and $y'(-1) = y'(1)$ (periodic SL problem)

Eigenvalues and eigenfunctions:

- $\lambda = 0$ is an eigenvalue with eigenfunction $y_0(x) = 1$.
- $\lambda = (k\pi)^2$ with eigenfunctions $\cos(k\pi x)$ and $\sin(k\pi x)$ with $k \in \mathbb{Z}^+$.

The expansion of a function f on the interval $[-1,\ 1]$ is just the regular Fourier series. Since $f(x) = 1$ is already an eigenfunction, then it is equal to its Fourier series (i.e. $a_0/2 = 1$ and $a_n = b_n = 0$ for $n \geq 1$).

The function $g(x) = x$ is odd, then $a_n = 0$ for all n and

$$b_n = 2\int_0^1 x\sin(n\pi x)dx = 2\frac{(-1)^{n+1}}{n}$$

Hence

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)\,.$$

Exercise 4. $y'' + \lambda y = 0,\ 0 < x < 1,\ y(0) = y'(0)$ and $y(1) = y'(1)$

Eigenvalues and eigenfunctions: Consider three cases

- Case $\lambda < 0$: Set $\lambda = -\nu^2$ with $\nu > 0$. The general solution of the DE is $y(x) = C_1 e^{\nu x} + C_2 e^{-\nu x}$ and $y'(x) = \nu(C_1 e^{\nu x} - C_2 e^{-\nu x})$. The condition $y(0) = y'(0)$ leads to $C_2(\nu + 1) = C_1(\nu - 1)$ and the condition $y(1) = y'(1)$ to the condition $C_2(\nu + 1)e^{-\nu} = C_1(\nu - 1)e^{\nu}$. It follows that $C_2(\nu + 1)e^{-\nu} = C_2(\nu + 1)e^{\nu}$ and then $C_2 = 0$. This system reduces to $C_1(\nu - 1) =$. If $C_1 \neq 0$, then $\nu = 1$ and we have nontrivial solution. Thus $\lambda_0 = -1$ is an eigenvalue with eigenfunction $y_0(x) = e^x$.
- Case $\lambda = 0$: The general solution of the DE is $y(x) = Ax + B$. The condition $y(0) = y'(0)$ gives $A = B$. Then the condition $y(1) = y'(1)$ gives $2A = A$ and so $A = B = 0$ and $\lambda = 0$ is not an eigenvalue.
- Case $\lambda > 0$: Set $\lambda = \nu^2$ with $\nu > 0$. The general solution of the DE is $y(x) = C_1\cos(\nu x) + C_2\sin(\nu x)$ and $y'(x) = \nu(-C_1\sin(\nu x) + C_2\cos(\nu x))$. The condition $y(0) = y'(0)$ leads to $C_1 = \nu C_2$ and the condition $y(1) = y'(1)$ leads to $C_1\cos\nu + C_2\sin\nu = -\nu C_1\sin\nu + \nu C_2\cos\nu$. After eliminating C_1 in the system we get $C_2(1 + \nu^2)\sin\nu = 0$. If $C_2 = 0$, then $C_1 = 0$ and the solution is trivial. In order to get a nontrivial solution, we need $C_2 \neq 0$, then $\sin\nu = 0$ and $\nu = k\pi$ with $k \in \mathbb{Z}^+$. The eigenvalues are then $\lambda_k = \nu_k^2 = (k\pi)^2$ with corresponding eigenfunction $y_k(x) = \nu_k\cos(\nu_k x) + \sin(\nu_k x)$.

Norms of eigenfunctions:

- For eigenvalue $\lambda_0 = -1$, eigenfunction $y_0(x) = e^x$

$$\|y_0\|^2 = \int_0^1 e^{2x}dx = \frac{e^2 - 1}{2}$$

- For eigenvalue $\lambda_k = \nu_k^2 = (k\pi)^2$, eigenfunction $y_k(x) = \nu_k\cos(\nu_k x) + \sin(\nu_k x)$

$$\|y_k\|^2 = \int_0^1 [\nu_k\cos(\nu_k x) + \sin(\nu_k x)]^2 dx = \frac{1 + \nu_k^2}{2}.$$

Expansion of $f(x)=1$: We have $1=c_0y_0(x)+\sum_{k=1}^{\infty}c_ky_k(x)$ with $c_j=\dfrac{<1,y_j>}{\|y_j\|^2}$. We have

$$<1,y_0>=\int_0^1 \mathrm{e}^x dx=\mathrm{e}-1 \quad \text{and} \quad c_0=2\frac{\mathrm{e}-1}{\mathrm{e}^2-1}=\frac{2}{\mathrm{e}+1}$$

For $k\geq 1$, we have

$$<1,y_k>=\int_0^1[\nu_k\cos(\nu_kx)+\sin(\nu_kx)]dx=\frac{1-(-1)^k}{\nu_k} \quad \text{and} \quad c_k=2\frac{1-(-1)^k}{\nu_k(1+\nu_k^2)}$$

Therefore

$$1=\frac{2\mathrm{e}^x}{\mathrm{e}+1}+2\sum_{k=1}^{\infty}\frac{1-(-1)^k}{\nu_k(1+\nu_k^2)}[\nu_k\cos(\nu_kx)+\sin(\nu_kx)]$$

Expansion of $g(x)=x$: We have $x=c_0y_0(x)+\sum_{k=1}^{\infty}c_ky_k(x)$ with $c_j=\dfrac{<x,y_j>}{\|y_j\|^2}$. We have

$$<x,y_0>=\int_0^1 x\mathrm{e}^x dx=[x\mathrm{e}^x-\mathrm{e}^x]_0^1=1 \quad \text{and} \quad c_0=\frac{2}{\mathrm{e}^2-1}$$

For $k\geq 1$, we have

$$<x,y_k>=\int_0^1 x[\nu_k\cos(\nu_kx)+\sin(\nu_kx)]dx=\frac{2(-1)^k-1}{\nu_k} \quad \text{and} \quad c_k=2\frac{2(-1)^k-1}{\nu_k(1+\nu_k^2)}$$

Therefore

$$x=\frac{2\mathrm{e}^x}{\mathrm{e}^2-1}+2\sum_{k=1}^{\infty}\frac{2(-1)^k-1}{\nu_k(1+\nu_k^2)}[\nu_k\cos(\nu_kx)+\sin(\nu_kx)]$$

Exercise 6. Same questions as in Exercise 5 for the SL-problem

$$x^2y''+xy'+\lambda y=0, \quad 1<x<L, \quad y'(1)=0, \quad y'(L)=0,$$

(1) Adjoint form of the DE: $(xy')'+\dfrac{\lambda}{x}y=0$.
(2) The weight associated with the SL-problem is $r(x)=\dfrac{1}{x}$ and the inner product is defined by
$$\langle f,g\rangle_r=\int_1^L f(x)g(x)\frac{1}{x}\,dx.$$
(3) Note that the DE is Cauchy-Euler with characteristic equation $m^2+\lambda=0$. Consider 3 cases.
   - If $\lambda<0$, set $\lambda=-\nu^2$ with $\nu>0$. The general solution of the DE is $y(x)=Ax^{\nu}+Bx^{-\nu}$ and we have $y'(x)=\nu Ax^{\nu-1}-\nu Bx^{-\nu-1}$. The condition $y'(1)=0$ and $y'(L)=0$ imply $A-B=0$ and $\nu(AL^{\nu-1}-BL^{-\nu-1})=0$ since $L>1$, $\nu>0$, then the only solution is $A=B=0$ and $\lambda<0$ cannot be an eigenvalue.
   - If $\lambda=0$. The general solution of the DE is $y(x)=A+B\ln x$ and $y'(x)=B/x$. The condition $y'(1)=0$ and $y'(L)=0$ imply $B=0$ and A arbitrary. Therefore $\lambda=0$ is an eigenvalue with eigenfunction $y_0(x)=1$.
   - If $\lambda>0$, set $\lambda=\nu^2$ with $\nu>0$. The general solution of the DE is $y(x)=A\cos(\nu\ln x)+B\sin(\nu\ln x)$ and
   $$y'(x)=\frac{\nu}{x}[-A\sin(\nu\ln x)+B\cos(\nu\ln x)]\,.$$

The condition $y'(1) = 0$ gives $B = 0$. Then $y'(L) = 0$ implies $\frac{\nu A}{L}\sin(\nu \ln L) = 0$. To obtain y nontrivial, we need $A \neq 0$ and then $\sin(\nu \ln L) = 0$. Therefore $\nu \ln L = n\pi$ with $n \in \mathbb{Z}^+$. In this case the eigenvalues and eigenfunctions are:

$$\lambda_n = \nu_n^2 = \left(\frac{n\pi}{\ln L}\right)^2, \quad y_n(x) = \cos(\nu_n \ln x) = \cos\left(n\pi \frac{\ln x}{\ln L}\right), \quad n \in \mathbb{Z}^+$$

(4) The norms of the eigenfunctions are

$$\|y_0\|^2 = \langle y_0, y_0\rangle_r \int_1^L \frac{dx}{x} = \ln L$$

For $n \geq 1$

$$\|y_n\|^2 = \langle y_n, y_n\rangle_r = \int_1^L \cos^2(\nu_n \ln x)\frac{dx}{x} = \int_1^L \cos^2\left(n\pi \frac{\ln x}{\ln L}\right)\frac{dx}{x}.$$

To compute the integral, we use the substitution $t = \ln x$ so that $dt = \frac{dx}{x}$ and obtain

$$\|y_n\|^2 = \int_0^{\ln L} \cos^2\left(\frac{n\pi}{\ln L}t\right) dt = \frac{\ln L}{2}$$

- **Expansion of $f(x) = 1$ in y_n's:** Since $y_0(x) = 1$ is already an element of the orthogonal basis then we have $1 = c_0 + \sum_{n=1}^{\infty} c_n y_n(x)$ with $c_0 = 1$ and $c_n = 0$ for $n \geq 1$.
- **Expansion of $g(x) = x$ in y_n's:** We have $x = c_0 + \sum_{n=1}^{\infty} c_n y_n(x)$ with

$$c_0 = \frac{\langle x, y_0\rangle_r}{\|y_0\|^2} = \frac{1}{\ln L}\langle x, y_0\rangle_r = \frac{2}{\ln L}\int_1^L dx = \frac{L-1}{\ln L}$$

and for $n \geq 1$

$$\begin{aligned} c_n &= \frac{\langle x, y_n\rangle_r}{\|y_n\|^2} = \frac{2}{\ln L}\langle x, y_n\rangle_r = \frac{2}{\ln L}\int_1^L x\cos(\nu_n \ln x)\frac{dx}{x} \\ &= \frac{2}{\ln L}\int_0^{\ln L} \mathrm{e}^t \cos\left(\frac{n\pi}{\ln L}t\right) dt \end{aligned}$$

Since $\int \mathrm{e}^t \cos(at)dt = \frac{\mathrm{e}^t\left[\cos(at) + a\sin(at)\right]}{1+a^2} + C$, then

$$c_n = \frac{2L}{\ln L}\,\frac{(-1)^n - 1}{1+\nu_n^2}$$

We have

$$x = \frac{L-1}{\ln L} + \frac{2L}{\ln L}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{1+\nu_n^2}\cos(\nu_n \ln x)\ .$$

Exercise 8. Solve the BVP

$$\begin{array}{ll} u_t = u_{xx} & 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0 & t > 0 \\ u(\pi,t) = u_x(\pi,t) & t > 0 \\ u(x,0) = 1 & 0 < x < \pi \end{array}$$

We proceed by finding solutions with separated variables $u(x,t) = X(x)T(t)$ of the homogeneous part. This leads to the following ODE problems for X and T, where λ is the separation constant:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = 0, \quad X(\pi) = X'(\pi) \end{cases}, \qquad T'(t) + \lambda T(t) = 0\,.$$

To find the eigenvalues and eigenfunctions of the X-problem, we consider 3 cases

- Case $\lambda = -\nu^2$ with $\nu > 0$. In this case the general solution of the ODE is $X(x) = A\mathrm{e}^{\nu x} + B\mathrm{e}^{-\nu x}$ and $X'(x) = \nu A\mathrm{e}^{\nu x} - \nu B\mathrm{e}^{-\nu x}$. The condition $X'(0) = 0$ leads to $A - B = 0$ (or $A = B$) and then the condition $X(\pi) = X'(\pi)$ leads to $A(\mathrm{e}^{\nu\pi} + \mathrm{e}^{-\nu\pi}) = A\nu(\mathrm{e}^{\nu\pi} - \mathrm{e}^{-\nu\pi})$ if $A \neq 0$, then ν must satisfy $(\mathrm{e}^{\nu\pi} + \mathrm{e}^{-\nu\pi}) = \nu(\mathrm{e}^{\nu\pi} - \mathrm{e}^{-\nu\pi})$ or equivalently $\mathrm{e}^{2\pi\nu} = \dfrac{\nu+1}{\nu-1}$. This equation has a unique positive solution $\nu_0 \in (1, \ 2)$ (see figure) Hence $\lambda_0 = -\nu_0^2$ is

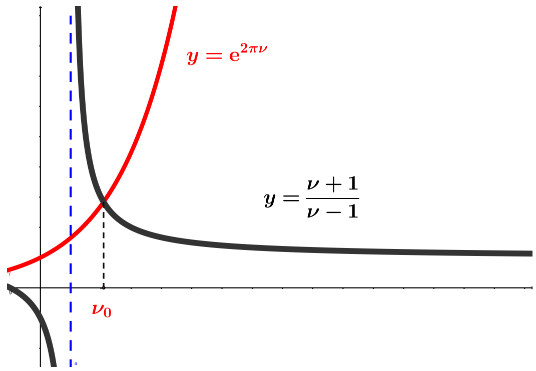

FIGURE 1. *Positive solutions of* $\mathrm{e}^{2\pi\nu} = \frac{\nu+1}{\nu-1}$

  an eigenvalue with corresponding eigenfunction $X_0(x) = \cosh(\nu_0 x)$.
- Case $\lambda = 0$. It is easily verified that 0 is not an eigenvalue.
- Case $\lambda = \nu^2$ with $\nu > 0$. The general solution of the ODE is $X(x) = A\cos(\nu x) + B\sin(\nu x)$. The condition $X'(0) = 0$ implies $B = 0$. Then the condition $X(\pi) = X'(\pi) = 0$ leads to $A\cos(\nu\pi) = -A\nu\sin(\nu\pi)) = 0$. For $A \neq 0$, ν needs to satisfy $\tan(\nu\pi) = \dfrac{-1}{\nu}$. This equation has infinitely many solutions, for every $n \in \mathbb{Z}^+$, the equation has a unique solution ν_n in the interval $\left(n - \dfrac{1}{2}, \ n\right)$ (see figure). The eigenvalues and eigenfunctions of the X-problem are:

$$\lambda_n = \nu_n^2, \quad \text{with} \quad \nu_n \in \left(n - \frac{1}{2}, \ n\right) \quad \tan(\nu_n\pi) = -\frac{1}{\nu_n}, \quad \text{and} \quad X_n(x) = \cos(\nu_n x).$$

The corresponding solutions of the T-problem are:

$$\begin{aligned} &\text{For} \quad \lambda_0 = -\nu_0^2, \quad T_0(t) = \mathrm{e}^{\nu_0^2 t}; \\ &\text{For} \quad \lambda_n = \nu_n^2, \quad T_n(t) = \mathrm{e}^{-\nu_n^2 t}, \quad n \in \mathbb{Z}^+\,. \end{aligned}$$

The solutions with separated variables of the homogeneous part of the BVP are

$$\mathrm{e}^{\nu_0^2 t}\cosh(\nu_0 x), \quad \mathrm{e}^{-\nu_n^2 t}\cos(\nu_n x), \quad n \in \mathbb{Z}^+\,.$$

The series representation of the general solution of the HP is

$$u(x,t) = c_0\mathrm{e}^{\nu_0^2 t}\cosh(\nu_0 x) + \sum_{n=1}^{\infty} c_n\mathrm{e}^{-\nu_n^2 t}\cos(\nu_n x)\,.$$

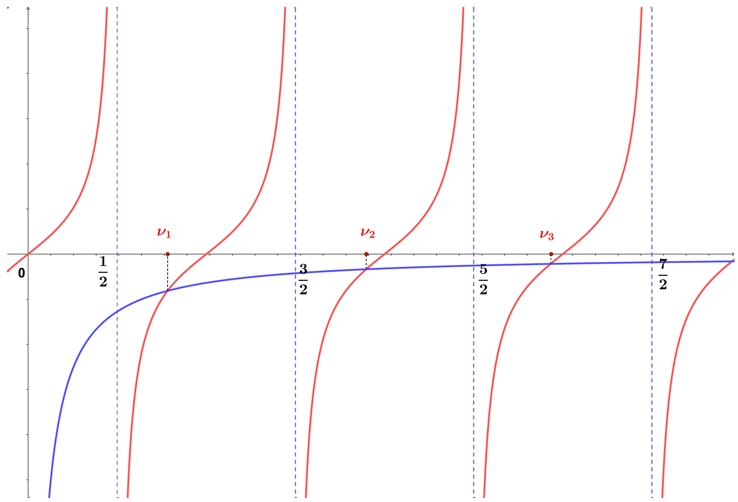

FIGURE 2. *Positive solutions of* $\tan(\nu\pi) = -\frac{1}{\nu}$

In order for u to satisfy the completed BVP, we need to have

$$u(x,0) = 1 = c_0 \cosh(\nu_0 x) + \sum_{n=1}^{\infty} c_n \cos(\nu_n x)\,.$$

The series is the generalized Fourier expansion of $\sin x$ in eigenfunctions of the X-problem. Thus

$$c_0 = \frac{\langle 1, \cosh(\nu_0 x)\rangle}{\|\cosh(\nu_0 x)\|^2} \quad \text{and} \quad c_n = \frac{\langle 1, \cos(\nu_n x)\rangle}{\|\cos(\nu_n x)\|^2} \quad \text{for } n \geq 1.$$

We have

$$\begin{aligned}
\|\cosh(\nu_0 x)\|^2 &= \int_0^{\pi} \cosh(\nu_0 x)\,dx = \frac{1}{2}\int_0^{\pi} (1 + \cosh(2\nu_0 x))\,dx \\
&= \frac{1}{2}\left(\pi + \frac{\sinh(2\nu_0\pi)}{2\nu_0}\right) = \frac{2\pi\nu_0 + \sinh^2(\nu_0\pi)}{4\nu_0}; \\
\|\cos(\nu_n x)\|^2 &= \int_0^{\pi} \cos^2(\nu_n x)\,dx = \frac{1}{2}\int_0^{\pi} (1 + \cos(2\nu_n x))\,dx \\
&= \frac{1}{2}\left(\pi + \frac{\sin(2\nu_n\pi)}{2\nu_n}\right) = \frac{\pi - \sin^2(\nu_n\pi)}{2}.
\end{aligned}$$

and

$$\begin{aligned}
\langle 1, \cosh(\nu_0 x)\rangle &= \int_0^{\pi} \cosh(\nu_0 x)\,dx = \frac{\sinh(\nu_0\pi)}{\nu_0} \\
\langle 1, \cos(\nu_n x)\rangle &= \int_0^{\pi} \cos(\nu_n x)\,dx = \frac{\sinh(\nu_n\pi)}{\nu_n}.
\end{aligned}$$

Hence

$$c_0 = \frac{4\sinh(\nu_0\pi)}{2\pi\nu_0 + \sinh(2\nu_0\pi)} \quad \text{and} \quad c_n = \frac{2\sin(\nu_n x)}{\nu_n(\pi - \sin^2(\nu_n\pi))}$$

and the solution of the BVP is

$$u(x,t) = \frac{4\sinh(\nu_0\pi)}{2\pi\nu_0 + \sinh(2\nu_0\pi)} e^{\nu_0^2 t}\cosh(\nu_0 x) + \sum_{n=1}^{\infty} \frac{2\sin(\nu_n x)}{\nu_n(\pi - \sin^2(\nu_n\pi))} e^{-\nu_n^2 t}\cos(\nu_n x)\,.$$

Exercise 10. Solve the BVP

$$\begin{array}{ll} u_{tt} = c^2 u_{xx} & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0 & t > 0 \\ u(\pi,t) - u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < \pi \\ u_t(x,0) = f(x) & 0 < x < \pi \end{array}$$

where

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < (\pi/2), \\ 1 & \text{if } (\pi/2) < x < \pi. \end{cases}$$

We proceed by finding solutions with separated variables $u(x,t) = X(x)T(t)$ of the homogeneous part. This leads to the following ODE problems for X and T, where λ is the separation constant:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ \\ X(0) = 0, \quad X(\pi) = X'(\pi) \end{cases}, \qquad \begin{cases} T''(t) + c^2 \lambda T(t) = 0\,, \\ \\ T(0) = 0 \end{cases}.$$

To find the eigenvalues of the X-problem, we consider three cases.

- $\lambda < 0$. Set $\lambda = -\mu^2$ with $\mu > 0$. In this case the general solution of the ODE is $X(x) = A\cosh(\mu x) + B\sinh(\mu x)$. The condition $X(0) = 0$ implies $A = 0$, then the second condition leads to $B\sinh(\mu\pi) = B\mu\cosh(\mu\pi)$. For $B \neq 0$, the parameter μ must satisfy $\sinh(\mu\pi) = \mu\cosh(\mu\pi)$ or equivalently $\mathrm{e}^{2\mu\pi} = \dfrac{1+\mu}{1-\mu}$. This equation has a unique positive solution μ_0 with $\mu_0 \in (0,\ 1)$. In fact $\mu_0 \approx 0.996$. Hence $\lambda_0 = -\mu_0^2$ is an eigenvalue with corresponding eigenfunction $X_0(x) = \sinh(\mu_0 x)$.
- It can be verified that $\lambda = 0$ is not an eigenvalue.
- $\lambda > 0$. Set $\lambda = \nu^2$ with $\nu > 0$. The general solution of the ODE is $X(x) = A\cos(\nu x) + B\sin(\nu x)$. The condition $X(0) = 0$ implies $A = 0$. Then the condition $X(\pi) = X'(\pi)$ leads to $B\sin(\nu\pi) = B\nu\cos(\nu\pi)$. For $B \neq 0$, ν needs to satisfy $\sin(\nu\pi) = \nu\cos(\nu\pi)$ or $\tan(\nu\pi) = \nu$. This equation has infinitely many solutions, for every $n \in \mathbb{Z}^+$, the equation has a unique solution ν_n in the interval $\left(n,\ \dfrac{2n+1}{2}\right)$. The eigenvalues and eigenfunctions of the X-problem are $\lambda_n = \nu_n^2$ and the corresponding eigenfunction $X_n(x) = \sin(\nu_n x)$.

For the negative eigenvalue $\lambda_0 = -\mu_0^2$, the corresponding T-equation becomes $T'' - c^2\mu_0^2 T = 0$ with general solution $T(t) = A\cosh(c\mu_0 t) + B\sinh(c\mu_0 t)$. The condition $T(0) = 0$ implies $A = 0$. The solution of HP of the BVP with separated variables is

$$u_0(x,t) = \sinh(c\mu_0 t)\sinh(\mu_0 x).$$

For the positive eigenvalues $\lambda_n = \nu_n^2$, the corresponding T-equation becomes $T'' + c^2\nu_n^2 T = 0$ with general solution $T(t) = A\cos(c\nu_n t) + B\sin(c\nu_n t)$. The condition $T(0) = 0$ implies $A = 0$. The solution of HP of the BVP with separated variables is

$$u_n(x,t) = \sin(c\nu_n t)\sin(\nu_n x).$$

The series representation of the general solution of HP is therefore

$$u(x,t) = c_0\sinh(c\mu_0 t)\sinh(\mu_0 x) + \sum_{n=1}^{\infty} c_n \sin(c\nu_n t)\sin(\nu_n x).$$

Now we use the nonhomogeneous condition to find the constants c_n's so that u solves the complete BVP.

$$u_t(x,0) = f(x) = (c\mu_0)c_0 \sinh(\mu_0 x) + \sum_{n=1}^{\infty} (c\mu_n)c_n \sin(\nu_n x).$$

Therefore

$$(c\mu_0)c_0 = \frac{\langle f(x), \sinh(\mu_0 x)\rangle}{\|\sinh(\mu_0 x)\|^2} \quad \text{and} \quad (c\mu_n)c_n = \frac{\langle f(x), \sin(\nu_n x)\rangle}{\|\sin(\nu_n x)\|^2}.$$

We have

$$\begin{aligned}
\|\sinh(\mu_0 x)\|^2 &= \int_0^{\pi} \sinh(\mu_0 x)^2 dx = \frac{1}{2}\int_0^{\pi} \left[\cosh(2\mu_0 x) - 1\right] dx \\
&= \frac{\sinh(2\mu_0\pi)}{4\mu_0} - \frac{\pi}{2} = \frac{\cosh^2(\mu_0\pi) - \pi}{2}; \\
\|\sin(\nu_n x)\|^2 &= \int_0^{\pi} \sin(\nu_n x)^2 dx = \frac{1}{2}\int_0^{\pi} \left[1 - \cos(2\nu_n x)\right] dx \\
&= \frac{\pi}{2} - \frac{\sin(2\nu_n\pi)}{4\nu_n} = \frac{\pi - \cos^2(\nu_n\pi)}{2}; \\
\langle f(x), \sinh(\mu_0 x)\rangle &= \int_{\pi/2}^{\pi} \sinh(\mu_0 x) dx = \frac{\cosh(\mu_0\pi) - \cosh(\mu_0\pi/2)}{\mu_0} \\
\langle f(x), \sin(\mu_n x)\rangle &= \int_{\pi/2}^{\pi} \sin(\mu_n x) dx = \frac{\cos(\mu_n\pi/2) - \cos(\mu_n\pi)}{\mu_n}
\end{aligned}$$

Hence

$$c_0 = \frac{\cosh^2(\mu_0\pi) - \pi}{2c(\cosh(\mu_0\pi) - \cosh(\mu_0\pi/2))} \quad \text{and} \quad c_n = \frac{2(\cos(\mu_n\pi/2) - \cos(\mu_n\pi))}{c\mu_n^2(\pi - \cos^2(\mu_n\pi))}.$$

The solution of the BVP is:

$$u(x,t) = \frac{(\cosh^2(\mu_0\pi) - \pi)\sinh(c\mu_0 t)\sinh(\mu_0 x)}{2c(\cosh(\mu_0\pi) - \cosh(\mu_0\pi/2))} + \sum_{n=1}^{\infty} \frac{2(\cos(\mu_n\pi/2) - \cos(\mu_n\pi))\sin(c\mu_n t)\sin(\mu_n x)}{c\mu_n^2(\pi - \cos^2(\mu_n\pi))}.$$

Exercises from LN 10

Exercise 2.

$$\begin{array}{ll} u_t = u_{xx} + \mathrm{e}^{-x} & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0,\ u(\pi,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < \pi \end{array}$$

We use the eigenfunctions expansion of the SL-problem $X'' + \lambda X = 0,\ X(0) = X(\pi) = 0$. That is, seek $u(x,t)$ as

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(nx)\,.$$

where $c_n(t)$ are functions of t that need to be determined. Since Fourier sine series of e^{-x} over $[0,\ \ \pi]$ is

$$\mathrm{e}^{-x} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(1+(-1)^n\mathrm{e}^{-\pi})}{1+n^2}\,\sin(nx)$$

the PDE $u_t = u_{xx} + \mathrm{e}^{-x}$ can be rewritten as

$$\sum_{n=1}^{\infty} c_n'(t)\sin(nx) = -\sum_{n=1}^{\infty} n^2 c_n(t)\sin(nx) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(1+(-1)^n\mathrm{e}^{-\pi})}{1+n^2}\,\sin(nx)\,.$$

The initial condition $u(x,0) = 0$ implies that $c_n(0) = 0$ for all $n \geq 1$. It follows that for $n \geq 1$, the function $c_n(t)$ satisfies the first order linear ODE problem

$$c_n'(t) + n^2 c_n(t) = \frac{2n(1+(-1)^n\mathrm{e}^{-\pi})}{\pi(1+n^2)}\,, \quad c_n(0) = 0.$$

We use the method of undetermined coefficients to find

$$c_n(t) = \frac{2(1+(-1)^n\mathrm{e}^{-\pi})}{n\pi(1+n^2)}\left(1-\mathrm{e}^{-n^2t}\right)\,.$$

Therefore the solution of the BVP is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2(1+(-1)^n\mathrm{e}^{-\pi})}{n\pi(1+n^2)}\left(1-\mathrm{e}^{-n^2t}\right)\sin(nx)\,.$$

Exercise 4.

$$\begin{array}{ll} u_t = u_{xx} + 2t & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0,\ u(\pi,t) = 100 & t > 0 \\ u(x,0) = 0 & 0 < x < \pi \end{array}$$

First seek a steady state function $s(x)$ that satisfies the end points conditions. That is $s''(x) = 0$, $s(0) = 0$ and $s(\pi) = 100$. We find $s(x) = \dfrac{100x}{\pi}$.

Now let $v(x,t) = u(x,t) - s(x)$. In order for u to solve the BVP, the function v must solve

$$\begin{array}{ll} v_t = v_{xx} + 2t & 0 < x < \pi,\ t > 0 \\ v(0,t) = 0,\ v(\pi,t) = 0 & t > 0 \\ v(x,0) = -s(x) = -\dfrac{100x}{\pi} & 0 < x < \pi \end{array}$$

Seek a solution v in the form $v(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(nx)$ where the functions $c_n(t)$ are to be determined. The Fourier sine series of $2t$ and $-\dfrac{100x}{\pi}$ are

$$2t = \frac{4t}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin(nx) \quad \text{and} \quad -\frac{100x}{\pi} = 200\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx)\,.$$

For such v, the BVP can be written as

$$\sum_{n=1}^{\infty} c_n'(t)\sin(nx) = -\sum_{n=1}^{\infty} n^2 c_n(t)\sin(nx) + \frac{4t}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin(nx)$$

$$\sum_{n=1}^{\infty} c_n(0)\sin(nx) = 200\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx)$$

The function $c_n(t)$ satisfies the initial value problem

$$c_n'(t) + n^2 c_n(t) = \frac{4(1-(-1)^n)t}{n\pi}, \quad c_n(0) = \frac{200(-1)^n}{n}$$

The UC method applied to the DE $y'(t) + n^2 y(t) = \dfrac{4(1-(-1)^n)t}{n\pi}$ gives the general solution as

$$y = K\mathrm{e}^{-n^2 t} + \frac{4(1-(-1)^n)}{n^3\pi}\left(t - \frac{1}{n^2}\right).$$

The solution that satisfies the initial condition is obtained for

$$K = \frac{4(1-(-1)^n)}{\pi n^5} + \frac{200(-1)^n}{n}$$

The solution of the original BVP is

$$u(x,t) = s(x) + v(x,t) = \frac{100x}{\pi} + \sum_{n=1}^{\infty} c_n(t)\sin(nx).$$

Exercise 6.

$$\begin{array}{ll} u_{tt} = u_{xx} + \sin(2x) & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0,\ u(\pi,t) = 0 & t > 0 \\ u(x,0) = \sin x,\ u_t(x,0) = \sin(3x) & 0 < x < \pi \end{array}$$

Seek a solution in the form $u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(nx)$ with $c_n(t)$ function of t to be determined.
For such a function $u(x,t)$ the BVP becomes

$$\sum_{n=1}^{\infty}\left(c_n''(t) + n^2 c_n(t)\right)\sin(nx) = \sin(2x)$$

$$\sum_{n=1}^{\infty} c_n(0)\sin(nx) = \sin x$$

$$\sum_{n=1}^{\infty} c_n'(0)\sin(nx) = \sin(3x)$$

This implies that for $n \neq 1, 2, 3$ the function $c_n(t)$ satisfies

$$c_n''(t) + n^2 c_n(t) = 0\,, \quad c_n(0 = c_n'(0) = 0\,.$$

Hence $c_n(t) = 0$ for $n \neq 1, 2, 3$.

- For $n = 1$ we have

$$c_1''(t) + c_1(t) = 0, \quad c_1(0) = 1, \quad c_1'(0) = 0$$

  with solution $c_1(t) = \cos t$.

- For $n = 2$ we have
$$c_2''(t) + 4c_2(t) = 1, \quad c_2(0) = 0, \quad c_2'(0) = 0$$
with solution $c_2(t) = \dfrac{1-\cos(2t)}{4}$.
- For $n = 3$ we have
$$c_3''(t) + 9c_3(t) = 0, \quad c_3(0) = 0, \quad c_3'(0) = 1$$
with solution $c_3(t) = \dfrac{\sin(3t)}{3}$.

The solution of the BVP is
$$u(x,t) = \cos t \sin x + \frac{1-\cos(2t)}{4} \sin 2x + \frac{\sin(3t)}{3} \sin(3x)\,.$$

Exercise 10. Let $f(x, y) = 1$ on the square $[0,\ 1]^2$. Find
1. The Fourier cosine-cosine series of f.
2. The Fourier cosine-sine series of f.
3. The Fourier sine-sine series of f.
4. The Fourier sine-cosine series of f.

(1) Fourier cosine-cosine series: Since the function 1 is already an element of the basis, then the Fourier cosine-cosine series of 1 is just the function 1.

(2) Fourier cosine-sine series:
$$1 = \frac{1}{2}\sum_{m=1}^{\infty} B_{0,m} \sin(my) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm} \cos(nx)\, \sin(my)$$
with
$$B_{nm} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} xy \cos(nx) \sin(my)\, dx\, dy.$$
We have
$$B_{0m} = \frac{4}{\pi^2}\left(\int_0^{\pi} dx\right)\left(\int_0^{\pi} \sin(my)\, dy\right) = \frac{2(1-(-1)^m)}{m\pi}$$
and for $n, m \geq 1$, $B_{n,m} = 0$. Hence for $0 \leq x \leq 1,\ \ 0 \leq y \leq 1$, we have
$$1 = \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{1-(-1)^m}{m} \sin(m\pi y)$$

(3) Fourier sine-sine series:
$$1 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm} \sin(n\pi x)\, \sin(m\pi y)$$
with
$$B_{nm} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} \sin(n\pi x) \sin(m\pi y)\, dx\, dy.$$
$$B_{n,m} = \frac{4(1-(-1)^n)(1-(-1)^m)}{\pi^2 nm}\,.$$
Hence for $0 \leq x \leq 1,\ \ 0 \leq y \leq 1$, we have
$$1 = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{[1-(-1)^n]\,[1-(-1)^m]}{nm} \sin(n\pi x)\, \sin(m\pi y)$$

(4) Fourier sine-cosine series: For $0 \le x \le 1,\ \ 0 \le y \le 1$, we have

$$1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin(n\pi x)$$

Exercise 14.

$$\begin{array}{ll} u_{tt} = u_{xx} + u_{yy}, & 0 < x < \pi,\ 0 < y < \pi,\ t > 0 \\ u(0,y,t) = u(\pi,y,t) = 0, & 0 < y < \pi,\ t > 0 \\ u(x,0,t) = u(x,\pi,t) = 0, & 0 < x < \pi,\ t > 0 \\ u(x,y,0) = 0.05x(\pi - x)y(\pi - y) & 0 < x < \pi,\ 0 < y < \pi \\ u_t(x,y,0) = 0 & 0 < x < \pi,\ 0 < y < \pi\ . \end{array}$$

If $u(x,y,t) = X(x)Y(y)T(t)$ is a nontrivial solution the homogeneous part of the BVP, then the functions X, Y, and T solve the ODE problems:

$$\begin{cases} X''(x) + \alpha X(x) = 0, \\ \\ X(0) = 0,\ X(\pi) = 0 \end{cases} \quad \begin{cases} Y''(y) + \beta Y(y) = 0, \\ \\ Y(0) = 0,\ Y(\pi) = 0 \end{cases} \quad \begin{cases} T''(t) + \lambda T(t) = 0 \\ \\ T'(0) = 0 \end{cases}$$

where α, β, λ are separation constants and $\lambda = \alpha + \beta$.
The eigenvalues and eigenfunctions of the X-problem are:

$$\alpha_n = n^2, \quad X_n(x) = \sin(nx), \quad n = 1,\ 2,\ 3,\ \cdots$$

The eigenvalues and eigenfunctions of the Y-problem are:

$$\beta_m = m^2, \quad Y_m(y) = \sin(my), \quad m = 1,\ 2,\ 3,\ \cdots$$

For each pair of integers n, m, we have $\lambda_{nm} = \omega_{nm}^2$ with $\omega_{nm} = \sqrt{n^2 + m^2}$ and an independent solution of the T-problem is $T_{nm}(t) = \cos(\omega_{nm} t)$. The solutions with separated variables of the homogeneous part are

$$\cos(\omega_{nm} t)\ \sin(nx)\ \sin(my)\,.$$

The series representation of the general solution is

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \cos(\omega_{nm} t)\ \sin(nx)\ \sin(my)\,.$$

To find the constants C_{nm} so that u solves the complete BVP we use the nonhomogeneous condition and then evaluate at $t = 0$.

$$u_t(x,y,0) = 0.05x(\pi - x)y(\pi - y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{nm} C_{nm}\ \sin(nx)\ \sin(my)\,.$$

The last series is therefore the Fourier sine-sine series of the function $0.05x(\pi - x)y(\pi - y)$. We have

$$C_{nm} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} 0.05x(\pi - x)y(\pi - y)\ \sin(nx)\ \sin(my)\, dxdy$$

An integration by parts gives

$$C_{nm} = \frac{0.8}{\pi^2} \frac{[1-(-1)^n]\,[1-(-1)^m]}{n^3 m^3}$$

Therefore the solution of the BVP is

$$u(x,y,t) = \frac{0.8}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{[1-(-1)^n]\,[1-(-1)^m]}{n^3 m^3} \cos(\omega_{nm} t)\ \sin(nx)\ \sin(my)\,.$$