HW 1:

(2) Let $a, b \in \mathbb{R}$ with $a < b$. Need to show that there $x \in \mathbb{R} \setminus \mathbb{Q}$ with $a < x < b$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then we can find $q_1, q_2 \in \mathbb{Q}$ such that $a \leq q_1 < q_2 \leq b$. Let $\varepsilon > 0$ with $\varepsilon \notin \mathbb{Q}$. Then exist $n \in \mathbb{N}$ s.t.

$$0 < \frac{\varepsilon}{n} < \frac{q_2 - q_1}{2}. \quad \text{(Archimedean Property)}$$

Let $x = q_1 + \frac{\varepsilon}{n}$, then $x \notin \mathbb{Q}$ (otherwise $(x - q_1)n = \varepsilon$ would be in $\mathbb{Q}$).

and $a < x < b$.

(3) Let $a_0 \in \mathbb{R}$. For $n \in \mathbb{N}$, let $a_n = a_0 - \frac{1}{n}$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $r_n \in \mathbb{Q} \cap (a_n, a_0)$.

Consider the set $E = \{ r_n, \ r_n \in \mathbb{Q} \cap (a_n, a_0), \ n \in \mathbb{N} \}$.

Then a_0 is an upper bound for E. and in fact

$a_0 = \sup E$.

Indeed, if $b < a_0$, let $N \in \mathbb{N}$ such that $\frac{1}{N} < a_0 - b$. Then $b < a_0 - \frac{1}{N} < r_N$. and $r_N \in E$. Hence b is not an upper bound for E.

(8) $2^{\mathbb{N}}$ is uncountable.

[In general we have the following the following

<u>Theorem</u> (Cantor) Let E be a set. Then there is no surjective function from E to 2^E

<u>Proof:</u> Let $f: E \longrightarrow 2^E$ be a function. Let $X = \{e \in E : e \notin f(e)\}$. Then $X \in 2^E$.

If f were surjective, then there would be $x \in E$ such that $X = f(x)$. This would mean that x ~~$\notin X$~~ if $x \notin X = f(x)$, then $x \in X$ by definition of X)

this is a contradiction and X has no preimage and f is not surjective.]

It follows that there is no surjective function from $\mathbb{N}$ to $2^{\mathbb{N}}$ and so $2^{\mathbb{N}}$ is uncountable.

$\mathbb{N}^{\mathbb{N}}$ is uncountable.

<u>Proof 1</u>: Let $E \in 2^{\mathbb{N}}$ (E subset of $\mathbb{N}$). We can find a function $f_E : \mathbb{N} \longrightarrow \mathbb{N}$ such that $f_E(\mathbb{N}) = E$.

[for example define $f_E(e) = e$ for $e \in E$ and

$f_E(n) = e_1$, for any $n \notin E$, where e_1 is the smallest element of

Consider the function $\Phi : 2^{\mathbb{N}} \longrightarrow \mathbb{N}^{\mathbb{N}}$: $\Phi(E) = f_E$.

Then Φ is one to one (because $f_E(\mathbb{N}) = E$ & $f_F(\mathbb{N}) = F$ are distinct if $E \neq F$).

Thus $2^{\mathbb{N}}$ is equipotent to $\Phi(2^{\mathbb{N}}) \subset \mathbb{N}^{\mathbb{N}}$

Hence $\mathbb{N}^{\mathbb{N}}$ is not countable

Proof 2: By contradiction. Suppose that $\mathbb{N}^{\mathbb{N}}$ is countable. we can write

$$\mathbb{N}^{\mathbb{N}} = \{f_1, f_2, \dots\}.$$

Define a function $g : \mathbb{N} \to \mathbb{N}$ as follows.

$g(1) \neq f_1(1)$ so that $g \neq f_1$

$g(2) \neq f_2(2)$ so that $g \neq f_2$ ....

$g(n) \neq f_n(n)$ so that $g \neq f_n$. for all n.

The function $g \in \mathbb{N}^{\mathbb{N}}$ but $g \neq f_n$ $\forall n$. Contradiction.

(10) $[0, 1)$ and $(0, 1)$ are equipotent.

Let $E = \{\frac{1}{2^n}, n \in \mathbb{N}\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$.

Define $f : (0, 1) \longrightarrow (0, 1]$ by $f(x) = \begin{cases} x & \text{if } x \notin E \\ \frac{1}{2^{n-1}} & \text{if } x \in E. \end{cases}$

Note that $f(E) = E \cup \{1\}$ and $f((0,1) \setminus E) = (0,1) \setminus E$

Hence $f(0,1) = (0, 1]$.

f is one to one on $(0, 1) \setminus E$ (it is the identity function)

and $f : E \to E \cup \{1\}$ is a bijection. Therefore

$f : (0, 1) \longrightarrow (0, 1]$ is a bijection.

$g : (0, 1] \longrightarrow [0, 1)$; $g(x) = 1 - x$ is a bijection.

(11) $(0,1)^2$ is equipotent to $(0,1)$

Define $f:(0,1)^2 \longrightarrow (0,1)$ as follows:

For $(x,y) \in (0,1)^2$, we use the decimal representation to express x and y as,

$$x = 0.x_1 x_2 x_3 \cdots \qquad (x_j, y_j \in \{0,1,\ldots,9\})$$
$$y = 0.y_1 y_2 y_3 \cdots$$

Define $f(x,y)$ by

$$f(x,y) = 0.x_1 y_1 x_2 y_2 \cdots$$

It can be verified that f is injective. Therefore,

$$(0,1)^2 \cong f[(0,1)^2] \subset (0,1).$$

Now we can define $g:(0,1) \longrightarrow (0,1)^2$ by $g(x) = (x, \frac{1}{2})$

g is injective. Therefore $(0,1) \cong g(0,1) \subset (0,1)^2$.

It follows that

$$(0,1) \cong (0,1)^2.$$

(16) Define a function $f: E \longrightarrow \mathbb{Q}$ as follows:

Let $x \in E$. Since E consists of only isolated points, then we can find $\delta_x > 0$ such that

$$(x-\delta_x, x+\delta_x) \cap E = \{x\}$$

and $(x-\delta_x, x+\delta_x) \cap (y-\delta_{\cancel{x}y}, y+\delta_y) = \phi$ if x, y are distinct points of E.

Let $q_x \in (x-\delta_x, x+\delta_x) \cap \mathbb{Q}$ (such q_x exists because $\mathbb{Q}$ is dense in $\mathbb{R}$). Set

$$f(x) = q_x$$

The function $f: E \to \mathbb{Q}$ thus defined is injective.

[if $x, y \in E$, $x \neq y$, then $(\cancel{x}-\delta_x, x+\delta_x) \cap (y-\delta_y, y+\delta_y) = \phi$ and so $q_x \neq q_y$].

We have then $E \cong f(E) \subset \mathbb{Q}$. Since $\mathbb{Q}$ is countable, then so is $f(E)$ and E.

(17) • E open $\iff \operatorname{int}(E)=E$

It follows from the definition of interior point that $\operatorname{int}(E)$ is an open subset of E. Now suppose E is open and $x\in E$, then there exists $r>0$ s.t. $(\cancel{x-r}, x+r)\subset E$. Thus $x\in \operatorname{Int}(E)$ $\therefore$ $\operatorname{Int}(E)\supseteq E$.

• E is dense $\iff \operatorname{int}(\mathbb{R}\setminus E)=\phi$

Suppose E is dense in $\mathbb{R}$. Let $x\in\mathbb{R}\setminus E$, then for every $\varepsilon>0$, $(x-\varepsilon, x+\varepsilon)\cap E\neq\phi$. Therefore $x\notin \operatorname{Int}(\mathbb{R}\setminus E)$ and $\operatorname{Int}(\mathbb{R}\setminus E)=\phi$.

Converseley, suppose $\operatorname{Int}(\mathbb{R}\setminus E)=\phi$. Let $x\in\mathbb{R}\setminus E$, Since $x\notin \operatorname{int}(\mathbb{R}\setminus E)$, then $\forall\varepsilon>0$, $(x-\varepsilon, x+\varepsilon)\cap E\neq\phi$

Therefore E is dense in $\mathbb{R}$.

HW2

① Let $a = \underline{\lim}\, a_n = \lim_{n\to\infty} \inf\{a_k, k \geq n\}$.

First we prove that a is a cluster point of the sequence.
Consider three cases

Case 1 $a = -\infty$.

For $A < 0$, $\exists N \in \mathbb{N}$ s.t. $\inf\{a_k; k \geq n\} < A$, $\forall n \geq N$.
Hence $\forall n \geq N$, there exists $k_n \geq n$ s.t. $a_{k_n} \leq A$.
The subsequence $a_{k_n} \longrightarrow -\infty$.

Case 2 $a = \infty$

For $A > 0$, $\exists N \in \mathbb{N}$ s.t. $\inf\{a_k; k \geq n\} > A$ $\forall n \geq N$.
in particular $a_n \geq A$ for $n > N$. In this case
$a_n \longrightarrow \infty$

Case 3 $a \in \mathbb{R}$.

Let $\varepsilon > 0$, $\exists N$ s.t. $\forall n > N$ we have
$$|\operatorname{Inf}\{a_k, k \geq n\} - a| < \varepsilon \quad \text{equivalently}$$
$$a - \varepsilon < \inf\{a_k; k \geq n\} < a + \varepsilon \qquad \forall n > N$$
~~Since~~ It follows that $a_k > a - \varepsilon$ for all $k \geq n$
and there exists $k_n \geq n$ s.t. $a_{k_n} < a + \varepsilon$.
Hence $a - \varepsilon < a_{k_n} < a + \varepsilon$
$a_{k_n} \longrightarrow a$. as $n \to \infty$.
$a \in \overline{\mathbb{R}}$ is a cluster pt for the sequence.

Next, we need to show that if b is any cluster pt of $\{a_n\}$, then $a \leq b$.

If $b \neq a$, then there exists a subsequence $\{a_{n_j}\}_j$ s.t. $a_{n_j} \longrightarrow b$ as $j \to \infty$. Hence
$$\operatorname{Inf}\{a_k, k \geq n_j\} \leq a_{n_j}$$
~~So~~ if $\varepsilon > 0$, $\exists J$; $a_{n_j} < b + \varepsilon$ for $j \geq J$
~~Therefore~~
$$\operatorname{Inf}\{a_k, k \geq n_j\} < b + \varepsilon \quad \text{for } j \geq J$$
This implies that $a \leq b$.

② "$\Longrightarrow$" Suppose that $a_n \to a$. Then every subsequence of $\{a_n\}$ converges to a and a is the only cluster point.

"$\Longleftarrow$" By contradiction, suppose that $\{a_n\}$ is a divergent sequence, we need to show that it has more than one cluster point. Consider the case $\{a_n\}$ bounded. Since $\{a_n\}$ diverges, then there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$, there exist $k_n > n$ and $l_n > n$ such that

$$|a_{k_n} - a_{l_n}| > \varepsilon_0 \quad (\{a_n\} \text{ does not satisfy Cauchy criterion})$$

The subsequences $\{a_{k_n}\}$ and $\{a_{l_n}\}$ have cluster points α, β.

Now we verify that $\alpha \neq \beta$. There are subsequences $\{a_{k_{n_j}}\}_j$ and $\{a_{l_{n_j}}\}$ of $\{a_{k_n}\}$ and $\{a_{l_n}\}$ such that

$$a_{k_{n_j}} \longrightarrow \alpha \quad \text{and} \quad a_{l_{n_j}} \longrightarrow \beta. \quad \text{as } j \to \infty.$$

Hence $\exists J$. s.t. if $j > J$, then $|\alpha - a_{k_{n_j}}| < \frac{\varepsilon_0}{4}$ and $|\beta - a_{l_{n_j}}| < \frac{\varepsilon_0}{4}$

Then

$$|\alpha - \beta| = \left|(\alpha - a_{k_{n_j}}) + (a_{k_{n_j}} - a_{l_{n_j}}) + (a_{l_{n_j}} - \beta)\right|$$

$$\geq |a_{k_{n_j}} - a_{l_{n_j}}| - |\alpha - a_{k_{n_j}}| - |\beta - a_{l_{n_j}}| > \varepsilon_0 - \frac{\varepsilon_0}{4} - \frac{\varepsilon_0}{4} = \frac{\varepsilon_0}{2}$$

The case $\{a_n\}$ unbounded can be proved in a similar way.

(6) For a real number y, denote by $[y]$ the largest integer $\leq y$. In particular, $[y]$ satisfies $y-1 < [y] \leq y$.

Let $x \in (0, 1)$. Define $a_1 \in \{0, \dots, p-1\}$ by $a_1 = [px]$.

Then $px - 1 < a_1 \leq px$. Hence

$$x_1 = px - a_1 \in (0, 1).$$

Repeat the construction for x_1:

Let $a_2 \in \{0, \dots, p-1\}$ given by $a_2 = [px_1]$.

then $px_1 - 1 < a_2 \leq px_1$ and $x_2 = px_1 - a_2 \in (0,1)$

Note that

$$x = \frac{a_1}{p} + \frac{x_1}{p} = \frac{a_1}{p} + \frac{1}{p}\left(\frac{a_2}{p} + \frac{x_2}{p}\right)$$

$$x = \frac{a_1}{p} + \frac{a_2}{p^2} + \frac{x_2}{p^2}.$$

By induction, suppose that we have integers

$a_1, \dots, a_n \in \{0, 1, \dots, p-1\}$ and $x_n \in (0, 1)$

such that

$$x = \frac{a_1}{p} + \frac{a_2}{p^2} + \dots + \frac{a_n}{p^n} + \frac{x_n}{p^n}$$

Define $a_{n+1} = [px_n] \in \{0, 1, \dots, p-1\}$. so that

$x_{n+1} = px_n - a_{n+1} \in (0, 1)$. and $x_n = \frac{a_{n+1}}{p} + \frac{x_{n+1}}{p}$

Hence

$$x = \frac{a_1}{p} + \dots + \frac{a_n}{p^n} + \frac{a_{n+1}}{p^{n+1}} + \frac{x_{n+1}}{p^{n+1}}$$

$$x = \sum_{j=1}^{n} \frac{a_j}{p^j} + \frac{x_n}{p^n}$$

We have then

$$0 \leq x - \sum_{j=1}^{n} \frac{a_j}{p^j} = \frac{x_n}{p^n} < \frac{1}{p^n}$$

Since $\frac{1}{p^n} \longrightarrow 0$ as $n \to \infty$, then

$$x = \sum_{j=1}^{\infty} \frac{a_j}{p^j}$$

Uniqueness: Suppose that $x=\sum_{j=1}^{\infty}\frac{a_j}{p^j}=\sum_{j=1}^{\infty}\frac{b_j}{p^j}$ and that $\exists k$ such that $a_k\neq b_k$.

Let N be the smallest such k. Thus $a_N\neq b_N$ and $a_j=b_j$ for $j<N$. We can assume $a_N>b_N\geq 0$.

It follows ~~that~~ (after canceling $\frac{a_j}{p^j}=\frac{b_j}{p^j}$ for $j<N$) that

$$\frac{a_N}{p^N}+\sum_{k=1}^{\infty}\frac{a_{N+k}}{p^{N+k}}=\frac{b_N}{p^N}+\sum_{k=1}^{\infty}\frac{b_{N+k}}{p^{N+k}}$$

and so

$$a_N-b_N=\sum_{k=1}^{\infty}\frac{b_{N+k}-a_{N+k}}{p^k}$$

Note that $a_N-b_N\geq 1$ and $|b_{N+k}-a_{N+k}|\leq p-1$

Hence $$\left|\sum_{k=1}^{\infty}\frac{b_{N+k}-a_{N+k}}{p^k}\right|\leq\sum_{k=1}^{\infty}\frac{p-1}{p^k}\leq 1.\leq a_N-b_N$$

Therefore $a_N=b_N+1$ and $\sum_{k=1}^{\infty}\frac{b_{N+k}-a_{N+k}}{p^k}=1$

Furthermore if there exists $k_0\geq 1$ such that $a_{N+k_0}\neq 0$, then $b_{N+k_0}-a_{N+k_0}<p-1$ and

$$\sum_{k=1}^{\infty}\frac{b_{N+k}-a_{N+k}}{p^k}<1 \quad \text{a contradiction}$$

Therefore $a_{N+k}=0$, $\forall k\geq 1$. and then $b_{N+k}=p-1$ $\forall k\geq 1$

Hence $$x=\sum_{j=1}^{N}\frac{a_j}{p^j}=\frac{\left(\sum_{j=1}^{N}a_jp^{N-j}\right)}{p^N}=\frac{q}{p^N}\quad,\ q<p^N$$

Also

$$x=\sum_{j=1}^{N}\frac{b_j}{p^j}+\sum_{j=1}^{\infty}\frac{p-1}{p^{N+j}}=\frac{q-1}{p^N}+\sum_{j=1}^{\infty}\frac{p-1}{p^{N+j}}$$

Conclusion: if $x\neq\frac{q}{p^N}$ for some $q<p^N$, then the decomposition is unique. If $x=\frac{q}{p^N}$, then x has two decompositions.

(7) The set $\mathbb{R}\setminus E$ is open. It can be written as a ~~this~~ countable disjoint union of open intervals

$$\mathbb{R}\setminus E = \bigcup_{k=1}^{\infty} (a_k, b_k)$$

with $(a_j, b_j) \cap (a_k, b_k) = \phi$ if $j \neq k$.

- If a_k, b_k are finite, then $a_k, b_k \in E$ and define $g : (a_k, b_k) \longrightarrow \mathbb{R}$ by.

$$g(x) = \frac{f(a_k) - f(b_k)}{a_k - b_k}(x - b_k) + f(b_k).$$

so that $g(a_k) = f(a_k)$ and $g(b_k) = f(b_k)$.

- If for some m we have $a_m = -\infty$ define $g : (-\infty, b_m) \longrightarrow \mathbb{R}$ by $g(x) = f(b_m)$
- If for some m we have $b_m = \infty$ define $g : (a_m, \infty) \longrightarrow \mathbb{R}$ by $g(x) = f(a_m)$.

The function g thus defined is a continuous extension of f to $\mathbb{R}$.

(8) <u>Claim:</u> The function f is continuous at 0 and at each irrational point but f is discontinuous at each nonzero rational number.

~~Let~~

• Continuity of f at $x=0$. We have $f(0)=0$. Let $\varepsilon>0$
if $x \in \mathbb{R}\setminus\mathbb{Q}$ and $|x|<\varepsilon$, then $|f(x)-f(0)|=|x|<\varepsilon$
If $x\in\mathbb{Q}$, $|x|=|\frac{p}{q}|<\varepsilon$, then $|f(x)-f(0)|=|p|\sin\frac{1}{|q|}\leq\frac{|p|}{|q|}=|x|<\varepsilon$.

• Now we prove that if $x\in\mathbb{R}\setminus\mathbb{Q}$, then for every $A>0$, there exists $\delta>0$ such that if $r\in(x-\delta,x+\delta)\cap\mathbb{Q}$, then $r=\frac{p}{q}$, $q>0$ with $q>A$

By contradiction suppose that there exists $A_0>0$ such that for every $\delta>0$, there exists $\frac{p_\delta}{q_\delta}\in(x-\delta,x+\delta)$, $p_\delta\in\mathbb{Z}$, $q_\delta\in\mathbb{N}$ and $q_\delta\leq A_0$. Take $\delta=\frac{1}{n}$, then $\frac{p_n}{q_n}\in(x-\frac{1}{n},x+\frac{1}{n})$ and $q_n\leq A_0$ for all n. It follows from

$$x-\frac{1}{n}<\frac{p_n}{q_n}<x+\frac{1}{n} \quad \text{that} \quad |p_n|\leq\max(|x-1|,|x+1|)\,q_n$$

Consider the set $E=\{\frac{p}{q}\in\mathbb{Q} \text{ s.t. } q\leq A_0,\ |p|\leq\max(|x-1|,|x+1|)A_0\}$
E is a finite set. Hence

$$\forall n\in\mathbb{N},\ \exists r_n\in E \text{ s.t. } |r_n-x|<\frac{1}{n}.$$

This together with the fact that E is finite implies that there exist $r\in E$ such that $|x-r|<\frac{1}{n}$ for all n. and so $x=r$
This a contradiction since x is irrational.

• f is discontinuous at $x_0\in\mathbb{Q}$, $x_0\neq0$. Write $x_0=\frac{p_0}{q_0}$, $q_0>0$
We have $|f(x_0)-x_0|=|p_0|\left|\sin\frac{1}{q_0}-\frac{1}{q_0}\right|=\varepsilon_0>0$
and $f(x_0)<x_0$ if $x_0>0$; $f(x_0)>x_0$ if $x_0<0$
Let $\delta>0$ arbitrary. If $x_0>0$ Let $x\in(x_0,x_0+\delta)$ ~~and~~
and x irrational. Then
$|f(x)-f(x_0)|=|x-f(x_0)|=|x-x_0+x_0-f(x_0)|>x_0-f(x_0)=\varepsilon_0$

If $x_0 < 0$, let $x \in (x_0 - \delta, x_0)$ and x_0 irrational

then $|f(x) - f(x_0)| = |f(x_0) - x_0 + x_0 - x| > f(x_0) - x_0 = \varepsilon_0$.

We have proved, that $\forall \delta > 0$, $\exists x \in (x_0 - \delta, x_0 + \delta)$ s.t.

$|f(x) - f(x_0)| > \varepsilon_0$. $\therefore$ f discontinuous at x_0.

- f is continuous at $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. Note that $f(x_0) = x_0$.

Let $\varepsilon > 0$, we need to find $\delta > 0$ s.t. if $x \in (x_0 - \delta, x_0 + \delta)$, then $|f(x) - f(x_0)| < \varepsilon$.

. If $x \in (x_0 - \delta, x_0 + \delta)$ and $x \notin \mathbb{Q}$, then $|f(x) - f(x_0)| = |x - x_0| < \delta$ in this case $|f(x) - f(x_0)| < \varepsilon$ provided $\delta \leq \varepsilon$.

The challenge is to find δ when $x \in \mathbb{Q}$..

Let $A > \sqrt{\dfrac{\max(|x_0 - 1|, |x_0 + 1|)}{3\varepsilon}}$ so that $\dfrac{1}{3A^2} < \dfrac{\varepsilon}{\max(|x_0 - 1|, |x_0 + 1|)}$.

For such A, there exists $\delta_A > 0$ such that

whenever $\dfrac{p}{q} \in (x_0 - \delta_A, x_0 + \delta_A)$ we have $q > A$ $\left(\begin{matrix} p \in \mathbb{Z} \\ q \in \mathbb{N} \end{matrix}\right)$

and so $|p| < \max(|x_0 - 1|, |x_0 + 1|)\, q$ $\left(\begin{matrix} \text{we can take} \\ \delta_A < 1 \end{matrix}\right)$

Let $\delta > 0$ be such that $\delta < \min(1, \frac{\varepsilon}{2}, \delta_A)$

For $x = \dfrac{p}{q} \in (x_0 - \delta, x_0 + \delta)$, we have, $q > A$

and $|f(x) - f(x_0)| = |f(x) - x + x - x_0| \leq |f(x) - x| + |x - x_0| \leq |f(x) - x| + \dfrac{\varepsilon}{2}$

since $|f(x) - x| = |p| \left|\sin\frac{1}{q} - \frac{1}{q}\right| \leq |p| \dfrac{1}{6q^3}$

$$\leq \frac{\max(|x_0 - 1|, |x_0 + 1|)\, q}{6q^3} \leq \frac{\max(|x_0 - 1|, |x_0 + 1|)}{6A^2}$$

$$\leq \frac{\max(|x_0 - 1|, |x_0 + 1|)}{2} \cdot \frac{\varepsilon}{\max(|x_0 - 1|, |x_0 + 1|)} = \frac{\varepsilon}{2}$$

Hence $|f(x) - f(x_0)| < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$.

This show that if $x \in (x_0 - \delta, x_0 + \delta)$, then $|f(x) - f(x_0)| < \varepsilon$ and so f is continuous at x_0.

(10) • A Lipschitz function is uniformly continuous

Suppose that $f: E \longrightarrow \mathbb{R}$ is Lipschitz with constant $L > 0$

Let $\varepsilon > 0$ and $\delta < \frac{\varepsilon}{L}$, then for $x, y \in E$ with $|x-y| < \delta$ we have $|f(x) - f(y)| < L\,|x-y| < L\delta < \varepsilon$ $\therefore$ f uniformly continuous

• There are uniformly continuous functions that are not Lipschitz.

Let $f: [0,1] \longrightarrow \mathbb{R}$ be defined by $f(x) = \sqrt{x}$

f is continuous. Since $[0,1]$ is closed and bounded, then f is uniformly continuous.

f is not Lipschitz. If f were Lipschitz with constant L then we would have $|f(x) - f(0)| \leq L\,|x-0|$. That is

$$\sqrt{x} \leq Lx \quad \forall x \in [0,1].$$

This would mean that $\sqrt{x} > \frac{1}{L}$ $\forall x (0, 1]$

a contradiction.

(11) Let $f: [a,b] \longrightarrow \mathbb{R}$ be continuous and $\varepsilon > 0$.

The function f is uniformly continuous. There exists $\delta > 0$ such that $\forall x, y \in [a,b]$, $|x-y| < \delta$, we have $|f(x) - f(y)| < \varepsilon/2$

We can assume $\delta = \frac{2(b-a)}{n}$ for some $n \in \mathbb{N}$.

Consider the subdivision of $[a,b]$ by points

$$x_0 = a, \quad x_1 = a + \frac{b-a}{n}, \ldots, x_j = a + j\,\frac{b-a}{n}, \ldots, x_n = b.$$

so that $x_{j+1} - x_j = \frac{b-a}{n} < \delta$ for $j = 0, \ldots, n-1$

Define $\varphi: [a,b] \longrightarrow \mathbb{R}$ by $\varphi(x) = \frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j}(x - x_j) + f(x_j)$ for $x \in [x_j, x_{j+1}]$. The function φ is piecewise linear

For any $x \in [a,b]$, there exists k, $x \in [x_k, x_{k+1}]$ and so

$$|\varphi(x) - f(x)| \leq |\varphi(x) - f(x_k)| + |f(x_k) - f(x)|$$

$$\leq |\varphi(x) - \varphi(x_k)| + \frac{\varepsilon}{2} \leq |\varphi(x_{k+1}) - \varphi(x_k)| + \frac{\varepsilon}{2}$$

$$\leq |f(x_{k+1}) - f(x_k)| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(19) Suppose that each function f_n of a sequence $\{f_n\}_n$ is defined on a set E and that the sequence converges uniformly to a function $f : E \to \mathbb{R}$. Suppose that each f_n is continuous, we need to prove that f is also continuous on E.

Let $\varepsilon > 0$. The uniform converges of f_n implies that there exists $N > 0$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$, $\forall x \in E$, $n \geq$
let $n > N$. and $x_0 \in E$. The continuity of f_n at x_0 implies that there exists $\delta > 0$ such that

$$|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3} \quad \text{for } x \in E \cap (x_0 - \delta, x_0 + \delta).$$

We have then

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence f is continuous at x_0.