HW3

#3. Let $A = \{x \in [0,1]; x \text{ is irrational}\}$.

$A \subset [0,1] \implies m^*(A) \leq m^*([0,1]) = 1$.

We have $[0,1) = A \cup B$, when $B = \mathbb{Q} \cap [0,1)$

Since $\mathbb{Q}$ is countable, then B is countable and $m^*(B) = 0$.

Therefore: $1 = m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(A)$.

Hence $m^*(A) = 1$

#7. For each point $x \in A$, define the open interval ~~I~~

$$I_x = \left(x - \frac{\alpha}{4}, \; x + \frac{\alpha}{4}\right).$$ and consider the set

$$U = \bigcup_{x \in A} I_x$$

Then U is open and $A \subset U$.

We have $U \cap B = \phi$. Indeed, if there is $y \in U \cap B$ then there would by $x \in A$ such that $y \in I_x$. ~~Bo~~ But then $|y - x| < \left|x + \frac{\alpha}{4} - \left(x - \frac{\alpha}{4}\right)\right| = \frac{\alpha}{2}$. A contradiction.

It follows that $B \subset U^c$

Since U is open, then it is measurable and it follows from the definition of measurable sets, that

$$m^*(A \cup B) = m^*((A \cup B) \cap U) + m^*((A \cup B) \cap U^c)$$

Since $A \subset U$, $B \subset U^c$, then

$$(A \cup B) \cap U = A \quad \text{and} \quad (A \cup B) \cap U^c = B.$$

This means $m^*(A \cup B) = m^*(A) + m^*(B)$.

\# 11. For $n \in \mathbb{N}$, let $E_n = E \cap [-n, n]$. Then E_n is bounded and $E = \bigcup_{n=1}^{\infty} E_n$. It follows that

$$m^*(E) \leq \sum_{n=1}^{\infty} m^*(E_n).$$

If each E_n has outer measure 0, then E would have outer measure 0. ~~Therefore~~ Since $m^*(E) > 0$ then there exists n s.t. $m^*(E_n) > 0$.

\# 15. Since E is not measurable, then $\exists \varepsilon_0 > 0$ such that $\forall$ V open $V \supset E$ we have

$$m^*(V \setminus E) \geq \varepsilon_0.$$

Let $\mathcal{U}$ be an open set such that $E \subset \mathcal{U}$ and

$$m^*(\mathcal{U}) \leq m^*(E) + \frac{\varepsilon_0}{2}.$$

[Such an open set exists by the definition of outer measure]

Hence $$m^*(\mathcal{U}) - m^*(E) \leq \frac{\varepsilon_0}{2}. \text{ and}$$

$$m^*(\mathcal{U} \setminus E) \geq \varepsilon_0$$

Therefore $$m^*(\mathcal{U} \setminus E) > m^*(\mathcal{U}) - m^*(E).$$

HW4

#2: Let $E \subset \mathbb{R}$ and C_E be a choice set for the rational equivalence relation in E. Then $E = \bigcup_{x \in C_E} [x]$, where $[x]$ denote the equivalence class of x:

$$[x] = \{ y \in E : \; y - x \in \mathbb{Q} \}$$

Hence $[x] \subset x + \mathbb{Q}$ and therefore $[x]$ is a countable set. If C_E were a countable set, then $E = \bigcup_{x \in C_E} [x]$ would be a countable set (as a countable union of countable set). It follows that if ~~E is counta~~ $m^*(E) = 0$.
Therefore $m^*(E) > 0 \implies C_E$ is uncountable.

#5: "$\Longrightarrow$" Let f be an increasing function I and f continuous, $x_0 \in I$.
Let $\{a_n\}_n$, $\{b_n\}_n$ be sequences in I such that
$a_n < x_0 < b_n$ for all n. and $a_n \to x_0$, $b_n \to x_0$.
The continuity of f implies that $\lim_{n\to\infty} (f(a_n) - f(x_0)) = \lim_{n\to\infty} (f(b_n) - f(x_0)) = 0$
therefore

$$\lim_{n\to\infty} (f(b_n) - f(a_n)) = \lim_{n\to\infty} (f(b_n) - f(x_0)) + \lim_{n\to\infty} (f(x_0) - f(a_n)) = 0$$

"$\Longleftarrow$" Suppose that f is increasing on I, and there exists sequences $\{a_n\}$, $\{b_n\}$ with $a_n < x_0 < b_n$ and $\lim_{n\to\infty} (f(b_n) - f(a_n)) = 0$. We need to show that f is continuous at x_0.

Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $f(b_n) - f(a_n) < \varepsilon/2$ for $n \geq N$

Let $\delta = \min(x_0 - a_N, \; b_N - x_0)$. For any $x \in (x_0 - \delta, \; x_0 + \delta)$ we have $a_N < x < b_N$ and so it follows from $f \uparrow$ that

$$f(x) - f(x_0) \leq f(b_N) - f(x_0) \leq f(b_N) - f(a_N) < \varepsilon/2 < \varepsilon$$

$$f(x) - f(x_0) \geq f(a_N) - f(x_0) \geq f(a_N) - f(b_N) > -\varepsilon/2 > -\varepsilon$$

Therefore $|f(x) - f(x_0)| < \varepsilon \quad \forall x \in (x_0 - \delta, \; x_0 + \delta)$
and f continuous at x_0.

#8: • Let $E \subset [a,b]$ be a set of measure 0. and let $\varepsilon > 0$.
It follows from the definition of outer measure that there exists a countable collection of disjoint open intervals $\{I_n\}$: such that $E \subset \bigcup_n I_n$ and
$\sum_n \ell(I_n) \le \frac{\varepsilon}{L}$, where L is the Lipschitz constant of f.

If $x, y \in I_n$, then $|f(x)-f(y)| < L|x-y| < L\,\ell(I_n)$.
Hence $m^*(f(I_n)) \le L\,\ell(I_n)$.

$E \subset \bigcup_n I_n \implies f(E) \subset \bigcup_n f(I_n)$ and so

$$m^*(f(E)) \le \sum_n m^*(f(I_n)) \le \sum_n L\,\ell(I_n) \le L\,\frac{\varepsilon}{L} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $m(f(E)) = 0$.

• First we show that if $C \subset [a,b]$ is a closed set then $f(C)$ is closed. Let $u \in \overline{f(C)}$, we need to show that $u \in f(C)$. Let $\{y_j\}_j$ be a sequence in $f(C)$ such that $y_j \to u$. Since $y_j \in f(C)$, then $\exists x_j \in C$; $y_j = f(x_j)$. Since C is closed and bounded, then there is a subsequence $\{x_{n_j}\}_j$ of $\{x_j\}$ such that $x_{n_j} \to x^* \in C$. Now we prove that $u = f(x^*)$. Let $\varepsilon > 0$, $\exists J \in \mathbb{N}$ s.t. $|x_{n_j} - x^*| < \varepsilon$ if $j > J$.
and $|y_{n_j} - u| < \varepsilon$ if $j > J$.

Then
$$|u - f(x^*)| \le |u - y_{n_j}| + |y_{n_j} - f(x^*)| < \varepsilon + |f(x_{n_j}) - f(x^*)|$$
$$< \varepsilon + L|x_{n_j} - x^*| < \varepsilon + L\varepsilon = (1+L)\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, then $u = f(x^*)$ and ~~$u \in f(C)$~~
This means $u \in f(C)$ ∴ $f(C)$ is a closed set.

Let $F \subset [a,b]$ be an F_σ-set. Then
$F = \bigcup_j F_j$, where $F_j \subset [a,b]$ is a closed set $\forall j \in \mathbb{N}$
then $f(F) = \bigcup_j f(F_j)$) is an F_σ-set.
$f(F_j)$ closed

• Let $A \subset [a,b]$ be a measurable set. We need to show that $f(A)$ is a measurable set.

Since A is measurable, then there exists an F_σ-set F with $F \subset A$ and $m(A \setminus F) = 0$. (i.e. $m(A) = m(F)$)

we have then

$$f(A) = f(F \cup (A \setminus F)) = f(F) \cup f(A \setminus F)$$

F is an F_σ-set $\Rightarrow f(F)$ is an F_σ-set

$A \setminus F$ has measure $0 \Rightarrow f(A \setminus F)$ has measure 0

$\therefore$ $f(A)$ is a measurable set (a union of 2 measurable s

#9: • Let F_n be the union of the 2^n closed intervals $F_n^1, \dots, F_n^{2^n}$ that remain at the n-th stage of deletion of open middle intervals of lengths $\alpha/3^n$. So

$$F_0 = [0,1], \quad F_1 = [0, \tfrac{1}{2} - \tfrac{\alpha}{6}] \cup [\tfrac{1}{2} + \tfrac{\alpha}{6}, 1] = F_1^1 \cup F_1^2, \dots$$

$$F_n = \bigcup_{j=1}^{2^n} F_n^j \quad . \quad \text{We have } F_{n+1} \subset F_n \quad \forall n.$$

Let a_n be the length of each interval F_n^j. It follows from the construction of the F_n^j's that

$$a_n = \frac{1}{2}\left[a_{n-1} - \frac{\alpha}{3^n}\right]$$

$$\left[\, a_1 = \frac{1}{2}\left(1 - \frac{\alpha}{3}\right), \quad a_2 = \frac{1}{2^2} - \frac{\alpha}{2^2}\left(1 - \left(\frac{2}{3}\right)^2\right), \dots \right]$$

By induction, we can establish that

$$a_n = \frac{1}{2^n} - \frac{\alpha}{2^n}\left(1 - \left(\frac{2}{3}\right)^n\right)$$

We have then $F_n = \bigcup_{j=1}^{2^n} F_n^j$ and since the F_n^j's are disjoint, then

$$m(F_n) = \sum_{j=1}^{2^n} m(F_n^j) = 2^n m(F_n^j) = 2^n a_n =$$

$$m(F_n) = 1 - \alpha\left(1 - \left(\frac{2}{3}\right)^n\right)$$

Since $F = \bigcap_{n=1}^{\infty} F_n$ and $\cancel{F_n}$ $F_n \subset F_{n-1}$

then $$m(F) = \lim_{n\to\infty} m(F_n) = \lim_{n\to\infty} \left(1 - \alpha\left(1 - \left(\tfrac{2}{3}\right)^n\right)\right) = 1 - \alpha.$$

- $[0,1]\setminus F$ is dense in $[0,1]$.

Let $x \in F$, we need to show that $\forall \varepsilon > 0$, $\exists y \in [0,1]\setminus F$ such that $|x-y| < \varepsilon$.

Let $N \in \mathbb{N}$ s.t. $\frac{1}{2^N} < \varepsilon$. Since $x \in F = \bigcap_n F_n$, then $x \in F_N = \bigcup_{j=1}^{2^N} F_N^j$. So there exists a unique $j \in \{1, \dots, 2^N\}$ such that $x \in F_N^j$.

We have $$\ell(F_N^j) = \frac{1}{2^N} - \frac{\alpha}{2^N}\left(1 - \left(\tfrac{2}{3}\right)^N\right) < \frac{1}{2^N} < \varepsilon$$

Let I_N^j be the open interval removed from F_N^j in the construction of F_{N+1}. If $y \in I_N^j$, then $y \notin F_{N+1}$ and so $y \notin F$.

$$x, y \in F_N^j \implies |x-y| < \ell(F_N^j) < \varepsilon.$$

So $[0,1]\setminus F$ is dense in $[0,1]$.

#12: A perfect set $X \subset \mathbb{R}$ is uncountable

By contradiction, suppose that X is countable. Set $X = \{x_j\}_{j \in \mathbb{N}}$.

Let $y_1 \in X$ s.t. $y_1 \neq x_1$. Since X is perfect, there exists a closed interval I_1 centered at y_1 such that $x_1 \notin I_1$. Let $y_2 \in \mathring{I}_1 \cap X$ ($\mathring{S}$ will denote the interior of the set S). be such that y_2 is the center of a closed interval I_2 with $I_2 \subset I_1$ and $x_2 \notin I_2$. Such y_2 and I_2 exist because $\mathring{I}_1 \cap X$ is infinite.

Now by induction, suppose that we have points $y_1, \dots, y_n$ and closed intervals ~~such that~~ $I_1, \dots, I_n$, such that

- $I_n \subset I_{n-1} \subset \dots \subset I_1$,
- y_j is the center of I_j, $y_j \in X$.
- and $x_j \notin I_j$ for $j = 1, \dots, n$.

Since $\mathring{I}_n \cap X$ is infinite, then there $y_{n+1} \in \mathring{I}_n \cap X$ and a closed interval I_{n+1} centered at y_{n+1} such that $x_{n+1} \notin I_{n+1}$.

We have then a nested collection $\{I_n\}_{n \in \mathbb{N}}$ of closed intervals each centered at $y_n \in X$.

Let $K = \bigcap_{n=1}^{\infty} I_n$, then $K \neq \phi$

and since $\forall x \in X$, $\exists n$; $x = x_n \notin I_n$, then $x \notin K$.

i.e. $X \cap K = \phi$. Now the sequence $\{y_n\}_n \subset X$ is bounded. and therefore $\{y_n\}$ has a convergent subsequence $\{y_{n_j}\}_j$ $y_{n_j} \to y^* \in X$, because X is closed.

but $y^* \in K$, since $y^* \in I_n$, $\forall n$. This is a contradiction