Fall 2022 - Real Analysis
Extra homework about l^p-spaces

The aim here is to describe some properties the l^p-spaces.

For $0 < p < \infty$, the space l^p consists of all infinite vectors (or sequences) $x = (x_1, x_2, \cdots) = (x_j)_{j=1}^{\infty}$ with $x_j \in \mathbb{R}$ such that $\sum_{j=1}^{\infty} |x_j|^p < \infty$. For $p = \infty$, the space l^∞ consists of all infinite vectors (or sequences) $x = (x_1, x_2, \cdots) = (x_j)_{j=1}^{\infty}$ with $x_j \in \mathbb{R}$ such that $\sup_{j\in\mathbb{N}} |x_j| < \infty$. For $0 < p \leq \infty$ define $||x||_p$ by

$$||x||_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \quad \text{if } 0 < p < \infty \quad \text{and} \quad ||x||_\infty = \sup_{j\in\mathbb{N}} |x_j| .$$

Exercise 1. Let $x = (1/j)_{j=1}^{\infty}$ and $y = (1)_{j=1}^{\infty}$

- Find all $p \in (0,\ \infty]$ such that $x \in l^p$. Same question for y.
- Show that if $0 < p < q \leq \infty$, then $l^p \subsetneq l^q$ and show that $||x||_q \leq ||x||_p$ for all $x \in L^p$.

Exercise 2. For $0 < p < \infty$ prove the following

- If $a,\ b \geq 0$, then $(a + b)^p \leq 2^p(a^p + b^p)$
- If $x,\ y \in l^p$, show that $||x + y||_p^p \leq 2^p(||x||_p^p + ||y||_p^p)$
- Prove that l^p is a linear (or vector) space. That is, for every $x, y \in L^p$ and $\alpha, \beta \in \mathbb{R}$, we have $\alpha x + \beta y \in l^p$.

The Hölder conjugate of $1 \leq p \leq \infty$ is the number q, $1 \leq q \leq \infty$, such that $\frac{1}{p} + \frac{1}{q} = 1$ (with the understanding that for $p = 1$ $q = \infty$). That is $q = \frac{p}{p-1}$.

Exercise 3. (**Hölder inequality for** $l^p,\ p \geq 1$)

- Let $0 < \sigma < 1$. Show that for $t \geq 0$ we have $t^\sigma \leq \sigma t + (1 - \sigma)$ and the equality holds iff $t = 1$.
- Let (p, q) be a Hölder pair of conjugate numbers with $1 < p < \infty$ and let $a, b \geq 0$. Show that $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ and the equality holds iff $b = a^{p-1}$ (*Hint*: apply the previous result to $t = \frac{a^p}{b^q}$ and $\sigma = 1/p$.)
- Let $1 \leq p \leq \infty$ and q be the Hölder conjugate of p. For $x = (x_j)_{j=1}^{\infty} \in l^p$ and $y = (y_j)_{j=1}^{\infty} \in l^q$, show that $xy = (x_j y_j)_{j=1}^{\infty} \in l^1$ and
$$||xy||_1 \leq ||x||_p\, ||y||_q .$$
Hint: First consider the case $p = 1$, $q = \infty$. For the case $p > 1$, $x \neq 0$ and $y \neq 0$, use the previous question to the unit vectors $u = \frac{x}{||x||_p} \in l^p$ and $v = \frac{y}{||y||_q} \in l^q$.

Exercise 4. (**Minkowski inequality for** $l^p,\ p \geq 1$)

- Show that if $x, y \in l^\infty$, then $||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$.
- Show that if $x, y \in l^1$, then $||x + y||_1 \leq ||x||_1 + ||y||_1$.
- Show that if $x, y \in l^p$, with $1 < p < \infty$, then $||x + y||_p \leq ||x||_p + ||y||_p$. *Hint*: You can use the decomposition $\sum_j |x_j + y_j|^p \leq A + B$ with $A = \sum_j |x_j|\,|x_j + y_j|^{p-1}$ and $B = \sum_j |y_j|\,|x_j + y_j|^{p-1}$. You can estimate A by using the Hölder inequality with $x \in l^p$ and $z = (|x_j + y_j|^{p-1})_j$ in l^q.
- Deduce that for $1 \leq p \leq \infty$, $(l^p, ||\,.\,||_p)$ is a normed vector space.

Exercise 5. Let $\{x^n\}_{n\in\mathbb{N}}$ be a sequence in l^p with $1 \leq p \leq \infty$. Thus for $n \in \mathbb{N}$, $x^n = (x_1^n, x_2^n, \cdots) = (x_j^n)_{j=1}^{\infty}$. The sequence x^n **is said to converge in** l^p **if there exists** $x = (x_j)_{j=1}^{\infty} \in l^p$ **such that** $\lim_{n\to\infty} ||x^n - x||_p = 0$.

- Show that if $x^n \longrightarrow x$ in l^p, then for every $j \in \mathbb{N}$ the sequence of real numbers $(x_j^n)_{n\in\mathbb{N}}$ converges to x_j.
- Consider the sequence $\{\delta^n\}_n$ in l^p given by $\delta_j^n = 0$ if $j \neq n$ and $\delta_n^n = 1$. That is, $\delta^n = (0, \cdots, 0, 1, 0, 0, \cdots)$, with 1 at the n-th coordinate. Find $\lim_{n\to\infty} \delta_j^n$. Is the sequence $(\delta^n)_n$ convergent in l^p?

Exercise 6. $(l^\infty, ||\,.\,||_\infty)$ **is a Banach space**. Let $\{x^n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in l^∞. That is, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, such that $||x^n - x^m||_\infty < \epsilon\ \ \forall n, m \geq N$.

- Show that for every $j \in \mathbb{N}$, the sequence of real numbers $\{x_j^n\}_n$ converges to a number $x_j \in \mathbb{R}$.

- Show that $x = \{x_j\}_j \in l^\infty$ and that $x^n \longrightarrow x$ in l^∞.

Exercise 7. Fatou's Lemma for Series. Let $u_j^n \geq 0$ for all $j, n \in \mathbb{N}$. Show that

$$\sum_{j=1}^{\infty} \liminf_{n\to\infty} u_j^n \;\leq\; \liminf_{n\to\infty} \sum_{j=1}^{\infty} u_j^n$$

Hint: If A and B are two subsets of $\mathbb{R}$, then $\inf(A) + \inf(B) \leq \inf(A+B)$.

Exercise 8. For $1 \leq p < \infty$, $(l^p, ||\,.\,||_p)$ is a Banach space. Let $\{x^n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in l^p with $1 \leq p \leq \infty$. That is, $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$, such that $||x^n - x^m||_p < \epsilon \;\; \forall n, m \geq N$.

- Show that for every $j \in \mathbb{N}$, the sequence of real numbers $\{x_j^n\}_n$ converges to a number $x_j \in \mathbb{R}$.
- Let $x = \{x_j\}_j$. Use Fatou's Lemma for series to prove that $||x - x^n||_p \;\leq\; \liminf\limits_{m\to\infty} ||x^m - x^n||_p$.
- Deduce that $x \in l^p$ and that $x^n \longrightarrow x$ in l^p.

Exercise 8. For $0 < p < 1$, $||\,.\,||_p$ is not a norm in l^p. Let $\delta^1 = (1, 0, 0, \cdots)$ and $\delta^2 = (0, 1, 0, 0, \cdots)$ be in l^p with $0 < p < 1$. Compute $||\delta^1||_p$, $||\delta^2||_p$ and $||\delta^1 + \delta^2||_p$. Deduce that the triangle inequality does not hold in l^p and that $||x||_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}$ is not a norm in l^p.

Exercise 9. Let $0 < p < 1$.

- Show that $(1+t)^p \leq 1 + t^p$ for all $t \geq 0$.
- Show that if $a, b \geq 0$, then $(a+b)^p \leq a^p + b^p$
- Deduce that if $x, y \in l^p$, then $||x+y||_p^p \leq ||x||_p^p + ||y||_p^p$.

Exercise 10. Metric in l^p for $0 < p < 1$. Consider the function

$$d_p : \, l^p \times l^p \longrightarrow [0, \; \infty), \quad d_p(x, y) = ||x - y||_p^p\,.$$

- Show that d_p is a distance function on l^p (i.e. $d_p(x, y) \geq 0$, $d_p(x, y) = 0$ iff $x = y$, and $d_p(x, y) \leq d_p(x, z) + d_p(z, y)$ for all $x, y, z \in l^p$).
- Show that (l^p, d_p) is a complete metric space (note that although (l^p, d_p) is a complete metric space, it is not a Banach space because the metric is not induced by a norm).

Exercise 11. Convexity of the unit ball. A set C in a linear metric space X is said to be convex if for every $x, y \in C$, we have $tx + (1-t)y \in C$ for all $t \in [0, \; 1]$.

- For $1 \leq p \leq \infty$, let $B = \{x \in l^p, ||x||_p < 1\}$. Show that B is a convex set of l^p.
- For $0 < p < 1$, let $B = \{x \in l^p, d_p(x, 0) < 1\}$. Show that B is not convex in l^p.