Class notes with exercises
for PHY 5346-7

Rajamani Narayanan

Department of Physics
Florida International University
Miami, FL 33199.

rajamani.narayanan@fiu.edu

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Preface

This is a one-year graduate course and it is assumed that all students in the course have an interest to work through the text book material on their own. We will begin by covering Chapters 1-5 in Landau’s Classical Theory of Fields. After this, we will discuss the topic of electromagnetic waves by covering Chapters 6 in Landau’s Classical Theory of Fields and referring to Jackson’s Classical Electrodynamics for reflection and refraction. We will then cover Sections I.1, I.2, I.3 (we will go into some detail while dealing with Section I.3 and use Jackson’s Classical Electrodynamics as a reference), II.6, II.7, III.21, III.22, IV.29 and IV.30 in Landau’s Electrodynamics of Continuous Media. We will return to Landau’s Classical Theory of Fields and work through Chapter 8 and 9.

These notes are a subset of the books suggested for the course and freely quotes from these books. As such, no claims about originality of these notes are made by the author.
Chapter 1

The principle of relativity of Einstein

- **Inertial frame**: A reference frame in which a freely moving body (a body that is not acted upon by external forces) proceeds with a constant velocity.
- There are arbitrarily many inertial frames. They move with a constant velocity with respect to each other.
- **Principle of relativity**: All laws of nature remain the same in all inertial frames.
- Velocity of propagation of interaction: Interactions in nature are not instantaneous and there is a finite velocity of the propagation of interaction.
- **Principle of relativity of Einstein**:
  - The velocity of propagation of interactions is the same in all inertial frames and this finite velocity is exactly equal to \( c = 299,792,458 \text{ m/sec} \). \( (1.1) \)
  - Physical motion has velocities less than or equal to \( c \).

1.1 Events and an invariant interval

An event is specified by its space and time coordinate. Motion is given by a continuous set of events,

\[
[ct, x(t), y(t), z(t)]
\]

in a reference frame.

Time cannot be absolute in relativistic mechanics. To see this, consider two events in one inertial frame, \( K \), that describe the propagation of interaction:

\[
(ct_i, x_i) \quad \text{and} \quad (ct_f, x_f). \tag{1.3}
\]

Consider another inertial frame, \( K' \), which is moving with respect to the first one with a velocity of \( v \) in the \( x \) direction. The two events in this new frame, if the time is absolute is given by

\[
(ct'_i, x'_i) = (ct_i, x_i - vt_i) \quad \text{and} \quad (ct'_f, x'_f) = (ct_f, x_f - vt_f). \tag{1.4}
\]

Since

\[
\frac{x_f - x_i}{t_f - t_i} \neq \frac{x'_f - x'_i}{t'_f - t'_i}, \tag{1.5}
\]

we conclude that time cannot be absolute.

Now consider two events in an inertial frame, \( K \), that describe the propagation of interaction in the most general case:

\[
(ct_i, x_i, y_i, z_i) \quad \text{and} \quad (ct_f, x_f, y_f, z_f). \tag{1.6}
\]

It follows that

\[
(x_f - x_i)^2 + (y_f - y_i)^2 + (z_f - z_i)^2 = c^2(t_f - t_i)^2. \tag{1.7}
\]
Consider the same two events in another inertial frame, \(K'\):

\[
(ct', x', y', z') \quad \text{and} \quad (ct', x'_f, y'_f, z'_f).
\]

(1.8)

It follows that

\[
(x'_f - x'_i)^2 + (y'_f - y'_i)^2 + (z'_f - z'_i)^2 = c^2(t'_f - t'_i)^2.
\]

(1.9)

Moving away from events that describe the propagation of interaction, let

\[
(c t_1, x_1, y_1, z_1) \quad \text{and} \quad (c(t_1 + dt_1), x_1 + dx_1, y_1 + dy_1, z_1 + dz_1)
\]

be two events that are separated infinitesimally in an inertial frame, \(K_1\). Let us define

\[
c^2 dt_1^2 = c^2 dt_1^2 - dx_1^2 - dy_1^2 - dz_1^2.
\]

(1.10)

If these two events describe a physical motion, then we know that

\[
\frac{\sqrt{dx_1^2 + dy_1^2 + dt_1^2}}{dt_1} \leq c \Rightarrow c^2 dt_1^2 \geq 0.
\]

(1.11)

Furthermore, \(cdt_1 = 0\) if the events describe the propagation of interaction. Let these same two infinitesimally separated events in another inertial frame, \(K_2\), be

\[
(c t_2, x_2, y_2, z_2) \quad \text{and} \quad (c(t_2 + dt_2), x_2 + dx_2, y_2 + dy_2, z_2 + dz_2)
\]

Then,

\[
c^2 dt_2^2 = c^2 dt_2^2 - dx_2^2 - dy_2^2 - dz_2^2.
\]

(1.12)

We know that \(c dt_1 = 0\) if \(c dt_2 = 0\) and \(c^2 dt_1^2 > 0\) if \(c^2 dt_2^2 > 0\). Therefore, we conclude that

\[
c^2 dt_1^2 = a c^2 dt_2^2.
\]

(1.13)

Assuming homogeneity of space and time, we conclude that \(a\) cannot depend upon \((x, y, z, t)\). Assuming isotropy of space, we conclude that \(a\) can only depend on the speed, \(v_{12}\), that relates the inertial frame, \(K_1\) to \(K_2\) and we explicitly show this dependence by writing,

\[
c^2 dt_1^2 = a(v_{12}) c^2 dt_2^2.
\]

(1.14)

If we have a third inertial frame, \(K_3\), then we have

\[
c^2 dt_1^2 = a(v_{13}) c^2 dt_3^2; \quad c^2 dt_2^2 = a(v_{23}) c^2 dt_3^2.
\]

(1.15)

It follows from Eq. (1.16) and Eq. (1.17) that

\[
a(v_{13}) = a(v_{12}) a(v_{23}).
\]

(1.16)

Since

\[
v_{13} = |\vec{v}_{13}| = |\vec{v}_{12} - \vec{v}_{23}|,
\]

(1.17)

it follows that the the left hand side of Eq. (1.18) depends on the angle between \(\vec{v}_{12}\) and \(\vec{v}_{23}\) but the right hand side does not. This can only happen if \(a\) is independent of the inertial reference frame and then Eq. (1.18) implies that \(a = 1\). Therefore, we come to the important conclusion that \(c^2 dt^2\) is the square of an invariant interval. Note that we can have two events for which \(c^2 dt^2 < 0\).

1.2 Timelike, Spacelike and Lightlike intervals

Let

\[
(ct_a, x_a, y_a, z_a) \quad \text{and} \quad (ct_b, x_b, y_b, z_b)
\]

denote two events in the inertial frame, \(K\), and

\[
(ct'_a, x'_a, y'_a, z'_a) \quad \text{and} \quad (ct'_b, x'_b, y'_b, z'_b)
\]

de note two events in the inertial frame, \(K'\).
1.3. Definition of a Moving Clock

denote the same two events in the inertial frame, $K'$. The time separating the two events in the two different inertial frames are

$$ t = |t_b - t_a|; \quad t' = |t'_b - t'_a|, \quad (1.22) $$

and the distance separating the two events in the two different inertial frames are

$$ \ell = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2}; \quad \ell' = \sqrt{(x'_b - x'_a)^2 + (y'_b - y'_a)^2 + (z'_b - z'_a)^2}. \quad (1.23) $$

The invariant finite interval is

$$ c^2\tau^2 = c^2t^2 - \ell^2 = c^2t'^2 - \ell'^2. \quad (1.24) $$

- **Timelike interval:** There exists a frame where the two events occur at the same location in space, namely, $\ell' = 0$, and this is only possible if

$$ c^2\tau^2 > 0 \Rightarrow c^2t^2 - \ell^2 > 0 \quad (1.25) $$

in all frames. In the special inertial frame where $\ell' = 0$, the elapsed time between the two events is

$$ t' = \tau = \frac{\sqrt{c^2t^2 - \ell^2}}{c}. \quad (1.26) $$

- **Spacelike interval:** There exists a frame where the two events occur at the same point in time, namely, $t' = 0$, and this is only possible if

$$ c^2\tau^2 < 0 \Rightarrow c^2t^2 - \ell^2 < 0 \quad (1.27) $$

in all frames. In the special inertial frame where $t' = 0$, the spatial separation between the two points is

$$ \ell' = \sqrt{-c^2\tau^2} = \sqrt{\ell^2 - c^2t^2}. \quad (1.28) $$

- **Lightlike interval:** If the two events describe the propagation of information from one location to another, then $\tau = 0$ and

$$ \ell = ct. \quad (1.29) $$

5 points: Problem 1.1: Show absolute future, absolute past and absolutely separated events in the $x - t$ plane. Explain your diagram in detail.

1.3 Definition of a moving clock

The path of a point particle in an inertial frame is labeled by $(ct(\tau), x(\tau), y(\tau), z(\tau))$. The path is not one with uniform velocity in general since there could be some force acting on this particle. The variable $\tau$ that labels the physical point on the path is defined using the invariant interval,

$$ \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2 = 1. \quad (1.30) $$

Since $c^2d\tau^2$ is an invariant interval, the path of the same point particle in another inertial frame will be labeled by $(ct'(\tau), x'(\tau), y'(\tau), z'(\tau))$ where $\tau$ labels the same physical point. The variable $\tau$ that labels the physical point of the particle along its path is the clock that moves with the particle.

1.4 Lifetime of a point particle

Consider a point particle with a finite lifetime that is moving under the experience of some force in an inertial frame. The path of the particle is given in one inertial frame by $(ct(\tau), x(\tau), y(\tau), z(\tau))$ for $\tau \in [0, T]$ where $0$ denotes the birth of the particle and the $T$ denotes the death of the particle. Note that the interval $T$ is invariant. Using Eq. (1.30), we can integrate over $\tau$ from 0 to $T$ noting that $t(0) = t_b$ and $t(T) = t_d$. The result of integration is

$$ T = \int_{t_b}^{t_d} \sqrt{(dt)^2 - \frac{1}{c^2} ((dx)^2 + (dy)^2 + (dz)^2)} = \int_{t_b}^{t_d} dt \sqrt{1 - \frac{1}{c^2} \left(v_x^2(t) + v_y^2(t) + v_z^2(t)\right)}. \quad (1.31) $$

Since the integrand is always less than or equal to unity, it follows that

$$ t_d - t_b \geq T, \quad (1.32) $$
with the equality holding if and only if the particle was at rest for the entire lifetime in the inertial frame of the observer. Therefore $T$ is the lifetime of the particle in its rest frame and the lifetime of the particle as measured by an observer in an inertial frame where the particle is moving is greater than $T$. This is often referred to as time dilation.

• **10 points: Problem 1.2:** An inertial observer sees a muon being born at rest (lifetime is 2.2 $\mu$s) and undergo a one dimensional motion with a constant acceleration of 9.8 m/s$^2$ for its entire lifetime. Do you think the observer has made a measurement that makes physical sense? If so, how long did the observer see the muon live? How far did the observer see the muon move in its entire lifetime? Is there an upper limit on the constant acceleration for the measurement to make physical sense? If so, what is the value of this limiting acceleration? How long will the observer see the muon live and how far will the observer see the muon move in its entire lifetime for the limiting acceleration?

### 1.5 Lorentz transformation

Let $K$ and $K'$ be both inertial frames and let $(ct, x, y, z)$ and $(ct', x', y', z')$ label the same event in $K$ and $K'$ respectively. A particle at rest for ever in $K$ frame moves with a velocity $v$ in the positive $x$ direction in the $K'$ frame for ever. Given this piece of information, we should be able to find the relation

$$
(ct, x, y, z) \rightarrow (ct' (ct, x, y, z), x'(ct, x, y, z), y'(ct, x, y, z), z'(ct, x, y, z))
$$

for all events.

Since a particle moving with a constant velocity in $K$ should also move with constant velocity in $K'$ the relation has to be linear. Furthermore, if the particle was moving with a constant velocity in the $x$ direction in $K$, it will move with a constant velocity in the $x$ direction in $K'$. Therefore, we can conclude that

$$
ct' = \Lambda_{00}ct + \Lambda_{10}x; \quad x' = \Lambda_{10}ct + \Lambda_{11}x; \quad y' = y; \quad z' = z,
$$

essentially decoupling the $y$ and $z$ coordinates. The quantities $\Lambda_{00}, \Lambda_{01}, \Lambda_{10}$ and $\Lambda_{11}$ can only depend on $v$ and $c$.

The infinitesimal interval has to be the same in both coordinates:

$$
c^2dt'^2 - dx'^2 = c^2dt^2 - dx^2 = (c\Lambda_{00}dt + \Lambda_{01}dx)^2 - (c\Lambda_{10}dt + \Lambda_{11}dx)^2.
$$

This results in the following conditions:

$$
\Lambda_{00}^2 - \Lambda_{10}^2 = 1; \quad \Lambda_{11}^2 - \Lambda_{01}^2 = 1; \quad \Lambda_{00}\Lambda_{01} = \Lambda_{10}\Lambda_{11}.
$$

A particle at rest at $x = 0$ in the $K$ frame has coordinates equal to

$$
t' = \Lambda_{00}t; \quad x' = c\Lambda_{10}t \Rightarrow x' = \frac{c\Lambda_{10}}{\Lambda_{00}}t'; \quad \Lambda_{00} > 0.
$$

Since this particle moves in the $K'$ frame with a velocity, $v$, in the positive $x$ direction, it follows that

$$
\beta \equiv \frac{v}{c} = \frac{\Lambda_{10}}{\Lambda_{00}} = \frac{\Lambda_{01}}{\Lambda_{11}},
$$

and we have used the third condition in Eq. (1.36) to obtain the last equality. Using the above relation and the first and second conditions in Eq. (1.36), we arrive at

$$
\Lambda_{00} = \Lambda_{11} = \frac{1}{\sqrt{1 - \beta^2}} \equiv \gamma
$$

and we have assume no reflection in the $x$ direction. The Lorentz transformation relating $K$ and $K'$ is

$$
\begin{pmatrix}
  ct' \\
  x' \\
  y' \\
  z'
\end{pmatrix}
= \begin{pmatrix}
  \gamma & \beta \gamma & 0 & 0 \\
  \beta \gamma & \gamma & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  ct \\
  x \\
  y \\
  z
\end{pmatrix}; \quad \Rightarrow
\begin{pmatrix}
  ct' \\
  x' \\
  y' \\
  z'
\end{pmatrix}
= \begin{pmatrix}
  \gamma & -\beta \gamma & 0 & 0 \\
  -\beta \gamma & \gamma & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  ct \\
  x \\
  y \\
  z
\end{pmatrix}
$$

Consider a particle moving a constant velocity,

$$
u = (u_x, u_y, u_z)
$$
1.6. RIGID BODY IN TROUBLE

starting at
\[ \mathbf{r}_0 = (x_0, y_0, z_0) \quad (1.42) \]
at \( t = 0 \) in the \( K \) frame. Therefore,
\[ x = x_0 + u_x t; \quad y = y_0 + u_y t; \quad z = z_0 + u_z t \quad (1.43) \]
describes the motion of the particle in the \( K \) frame. Using Eq. (1.40), we obtain
\[ t' = \gamma t + \frac{\beta \gamma}{c}(x_0 + u_x t); \quad x' = \beta \gamma c t + \gamma(x_0 + u_x t); \quad y' = y_0 + u_y t; \quad z' = z_0 + u_z t. \quad (1.44) \]
Eliminating \( t \) in favor of \( t' \) we find that the particle moves with a constant velocity,
\[ \mathbf{u}' = \frac{1}{1 + \frac{v u}{c^2}} \left( u_x + v, \quad u_y \sqrt{1 - \frac{v^2}{c^2}}, \quad u_z \sqrt{1 - \frac{v^2}{c^2}} \right), \quad (1.45) \]
starting at
\[ \mathbf{r}_0' = \left( \frac{x_0 \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v u}{c^2}}, \quad y_0 - \frac{v u}{c^2} x_0, \quad z_0 - \frac{v u}{c^2} x_0 \right). \quad (1.46) \]
at \( t' = 0 \) in the \( K' \) frame.

**10 points: Problem 1.3:** Prove that
\[ \mathbf{u} \cdot \mathbf{u} \leq c^2 \Rightarrow \mathbf{u}' \cdot \mathbf{u}' \leq c^2 \quad (1.47) \]
with equality implying equality.

**10 points: Problem 1.4:** Use \( g \) as the acceleration due to gravity for this problem. An inertial observer in the \( K \) frame sees a ball (you can assume it is a point ball!) bounce up and down on the ground (you can also assume that no kinetic energy is lost by the ball to the ground when it hits it). Assume the motion of the ball is entirely in the \( z \)-direction and assume that the ground is located at \( z = 0 \). Draw the motion in the \( z-t \) plane (use the horizontal axis for \( z \) and the vertical axis for \( t \)). \( K' \) is an inertial frame moving with respect to \( K \) in the positive \( z \) direction with a velocity \( v \). Derive the equation of motion for the ball in the \( K' \) frame and plots its motion in the \( z'-t' \) plane. Will this motion make physical sense independent of the maximum height reached by the ball in the \( K \) frame? If not, find the limiting height.

1.6 Rigid body in trouble

Consider a three-dimensional rigid cube of size \( L^3 \) at rest in the \( K \) frame with the center at the origin of the coordinate system. The rigid body occupies the space, \( -\frac{L}{2} \leq x, y, z \leq \frac{L}{2} \), for all time \( t \) in the \( K \) frame. Furthermore, \( \mathbf{u} = 0 \). It follows from Eq. (1.45) and Eq. (1.46) that
\[ \mathbf{u}' = (v, \quad 0, \quad 0); \quad \mathbf{r}_0' = \left( x_0 \sqrt{1 - \frac{v^2}{c^2}}, \quad y_0, \quad z_0 \right); \quad -\frac{L}{2} \leq x_0, y_0, z_0 \leq \frac{L}{2}. \quad (1.48) \]
The object in \( K' \) frame is moving with a velocity \( v \) in the \( x \) direction but it is no longer a cube of size \( L^3 \); instead it is a rectangular cuboid of size \( L \sqrt{1 - \frac{v^2}{c^2}} \) in the \( x \) direction and \( L^2 \) in the \( y \) and \( z \) directions. This is often referred to as length contraction.

**10 points: Problem 1.5:** Consider a three-dimensional cube of size \( L^3 \) moving with a velocity
\[ \mathbf{u} = (0, u, 0) \quad (1.49) \]
in the \( K \) frame with its center at the origin of the coordinate system at \( t = 0 \). Derive expressions for the shape and motion of the same object in the \( K' \) frame and describe its shape and motion in complete detail.

**10 points: Problem 1.6:** Consider an ideal cubic light source of size \( L^3 \) that is perfectly collimated and emitting light in the \( \hat{y} \) direction in the \( K \) frame. Derive expressions for the shape of the light source in the \( K' \) frame and the direction of the emitted light. Describe your answers in complete detail.
10 points: Problem 1.7: A one-dimensional string of tightly bound muons ($\mu$) occupy a length $L$ in the $x$-direction in their rest frame, $K$. Assume that the life time of tightly bound muons is $T_\mu$. Consider a frame, $K'$, that is moving with a velocity $v$ along the positive $x$-axis. Derive the life history of the one-dimensional string of tightly bound muon in the $K'$ frame. Draw a picture of the life history in the $x - t$ and the $x' - t'$ frame and describe what physically is seen in the two different frames.

1.7 Transformation of velocities

The Lorentz transformation in Eq. (1.40) related the coordinates in $K$ and $K'$ frame for any event. Consider a point particle that is moving in an arbitrary path in the $K$ frame given by $(ct(\tau), x(\tau), y(\tau), z(\tau))$ where $\tau$ labels the points along the path. The same path in the $K'$ frame is given by $(ct'(\tau), x'(\tau), y'(\tau), z'(\tau))$. Let us drop the dependence on $\tau$ in both frames for brevity. As per Eq. (1.40),

$$ct' = \gamma (ct + \beta x); \quad x' = \gamma (\beta ct + x); \quad y' = y; \quad z' = z. \tag{1.50}$$

Upon taking the derivatives with respect to $\tau$ on both sides, we obtain

$$\frac{dt'}{d\tau} = \gamma \left( \frac{dt}{d\tau} + \frac{v}{c^2} \frac{dx}{d\tau} \right); \quad \frac{dx'}{d\tau} = \gamma \left( \frac{v}{c^2} \frac{dt}{d\tau} + \frac{dx}{d\tau} \right); \quad \frac{dy'}{d\tau} = \frac{dy}{d\tau}; \quad \frac{dz'}{d\tau} = \frac{dz}{d\tau}. \tag{1.51}$$

Dividing the last three equations by the first equation results in

$$u'_x = \frac{v + u_x}{1 + \frac{v^2}{c^2}}; \quad u'_y = \frac{u_y}{1 + \frac{v^2}{c^2}}; \quad u'_z = \frac{u_z}{1 + \frac{v^2}{c^2}}. \tag{1.52}$$

where

$$u = \left( \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right); \quad u' = \left( \frac{dx'}{d\tau}, \frac{dy'}{d\tau}, \frac{dz'}{d\tau} \right). \tag{1.53}$$

are the instantaneous velocities of the particle in the $K$ and $K'$ frames respectively. Note that this formula looks like Eq. (1.45) but it is applicable for an arbitrary motion. Furthermore, Eq. (1.47) is valid at every instant.

We perform a Taylor expansion of Eq. (1.52) in $\beta$ to understand deviations from $u' = u + v\hat{x}$ valid in the limit $\beta \to 0$. We do not assume that $u$ itself is small compared to $c$. We first note that

$$\sqrt{1 - \beta^2} = 1 - \frac{1}{2}\beta^2 + \cdots; \quad \text{and} \quad \frac{1}{1 + \frac{v^2}{c^2}} = 1 - \frac{u_x}{c}\beta + \frac{u_x^2}{c^2}\beta^2 + \cdots. \tag{1.54}$$

Therefore,

$$u'_x = (v + u_x) \left( 1 - \frac{u_x}{c}\beta + \frac{u_x^2}{c^2}\beta^2 \right) + \cdots;$$

$$u'_y = u_y \left( 1 - \frac{u_x}{c}\beta + \frac{u_x^2}{c^2} - \frac{1}{2}\beta^2 \right) + \cdots;$$

$$u'_z = u_z \left( 1 - \frac{u_x}{c}\beta + \frac{u_x^2}{c^2} - \frac{1}{2}\beta^2 \right) + \cdots \tag{1.55}$$

It is useful to convert Eq. (1.52) into spherical polar coordinates:

$$u' \sin \theta' \cos \phi' = \frac{v + u \sin \theta \cos \phi}{1 + \frac{v^2}{c^2} \sin \theta \cos \phi}; \quad u' \sin \theta' \sin \phi' = \frac{u \sqrt{1 - \frac{v^2}{c^2} \sin \theta \sin \phi}}{1 + \frac{v^2}{c^2} \sin \theta \cos \phi}; \quad u' \cos \theta' = \frac{u \sqrt{1 - \frac{v^2}{c^2} \cos \theta}}{1 + \frac{v^2}{c^2} \sin \theta \cos \phi}$$

$$\Rightarrow \quad \tan \phi' = \frac{u \sqrt{1 - \frac{v^2}{c^2} \sin \theta \sin \phi}}{v + u \sin \theta \cos \phi}; \quad \tan \theta' \sin \phi' = \tan \theta \sin \phi. \tag{1.56}$$

Consider the case when $u' = u = c$. Then,

$$\sin \theta' \cos \phi' = \frac{\beta + \sin \theta \cos \phi}{1 + \sin \theta \cos \phi}; \quad \sin \theta' \sin \phi' = \frac{\sqrt{1 - \beta^2} \sin \theta \sin \phi}{1 + \beta \sin \theta \cos \phi}; \quad \cos \theta' = \frac{\sqrt{1 - \beta^2} \cos \theta}{1 + \beta \sin \theta \cos \phi}. \tag{1.57}$$

Specializing to the case, $\theta = \frac{\pi}{2}$, we obtain

$$\theta' = \frac{\pi}{2}; \quad \cos \phi' = \frac{\beta + \cos \phi}{1 + \beta \cos \phi}; \quad \sin \phi' = \frac{\sqrt{1 - \beta^2} \sin \phi}{1 + \beta \cos \phi}. \tag{1.58}$$
Assuming $\beta$ is small, we Taylor expand in $\beta$ and keep only the leading correction in $\beta$. We obtain

$$
\cos \phi' = \cos \phi + \beta \sin^2 \phi + \cdots; \quad \sin \phi' = \sin \phi - \beta \sin \phi \cos \phi + \cdots. 
$$

(1.59)

To solve the above equations, we assume a Taylor expansion of $\phi'$ in $\beta$ given by

$$
\phi' = \phi + c_1 \beta + \cdots.
$$

(1.60)

Inserting this into the two equations above, we have

$$
\cos \phi - c_1 \beta \sin \phi + \cdots = \cos \phi + \beta \sin^2 \phi + \cdots; \quad \sin \phi + c_1 \beta \cos \phi + \cdots = \sin \phi - \beta \sin \phi \cos \phi + \cdots.
\Rightarrow \quad c_1 = -\sin \phi.
$$

(1.61)

The small deviation, $\Delta \phi = \phi - \phi' = \beta \sin \phi$, is referred as the aberration of light.

- **10 points: Problem 1.8:** Derive the next term in the Taylor expansion in Eq. (1.60), namely, the term proportional to $\beta^2$.

- **10 points: Problem 1.9:** Provide one or more physical descriptions of a closed path in the $x - t$ plane given by

$$
x^2 + c^2 t^2 = a^2,
$$

(1.62)

with $a$ being a positive constant. Starting from Eq. (1.40) derive the path in the $x' - t'$ plane. You need to get the expression to a form that enables you to describe the path using analytical geometry. One or more plots of examples will not suffice since you need to provide all the required properties that classifies the path. Give one or more physical descriptions of the path in the $x' - t'$ plane.

- **10 points: Problem 1.10:** Consider a particle moving in the K frame with a velocity given by $u = u (\cos \theta i + \sin \theta j)$; $u > 0$ and $\theta \in [-\pi, \pi]$. Consider a frame K' that is moving in the negative $x$ direction with respect to K with a speed $v$. Let the velocity of the particle in the new frame be given by $u' = u' (\cos \theta' i + \sin \theta' j)$. Starting from Eq. (1.40) derive the following relations:

$$
\tan \theta' = \frac{\sin \theta \sqrt{1 - \frac{u^2}{c^2}}}{\cos \theta + \frac{u}{c}}; \quad 1 - \frac{u'^2}{c^2} = \frac{1 - \frac{u^2}{c^2}}{(1 + \frac{uc}{c^2})^2}.
$$

(1.63)

Quantitatively describe the behavior of $u'$ and $\theta'$ as a function of $\theta$ for several choices of $u$ and $v$. You may use examples but your end result has to be a complete analytical description for the entire range of $u$ and $v$.

- **10 points: Problem 1.11:** Starting from Eq. (1.40), derive the following formulas for the transformation of the components of acceleration:

$$
a_x' = \left(1 - \frac{\gamma^2}{c^2}\right) \frac{\gamma}{(1 + \frac{\gamma u}{c^2})} a_x;
$$

$$
a_y' = \frac{1 - \frac{\gamma^2}{c^2}}{(1 + \frac{\gamma u}{c^2})^2} a_y - \frac{\gamma u v}{c^2} \left(1 - \frac{\gamma^2}{c^2}\right) a_x;
$$

$$
a_z' = \frac{1 - \frac{\gamma^2}{c^2}}{(1 + \frac{\gamma u}{c^2})^2} a_z - \frac{\gamma u v}{c^2} \left(1 - \frac{\gamma^2}{c^2}\right) a_x.
$$

(1.64)

Discuss the physics of the acceleration in the K' frame for the special case where the particle is instantaneously at rest in the K frame and undergoes an acceleration of the form $a = a (\cos \theta i + \sin \theta j)$.

- **10 points: Problem 1.12:** An inertial frame K' has a constant velocity $v_1$ along the negative real axis corresponding to the inertial frame K. An inertial frame K'' has a constant velocity $v_2$ along the negative real axis corresponding to the inertial frame K'. Use Eq. (1.40) two times to directly show that K'' has a constant velocity

$$
v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}},
$$

(1.65)

along the negative real axis corresponding to the inertial frame K. Then explain why this makes sense with respect to formulas for the transformation of velocities from K to K'. You need to derive the formulas for the transformation of velocities.
Given the motion of a point particle, \( \dot{\mathbf{r}} \) and \( \ddot{\mathbf{r}} \) are the standard instantaneous velocity of the particle. We define the four position vector \( \mathbf{x} \), four velocity vector \( \mathbf{v} \) and the four acceleration \( \mathbf{a} \) respectively.

It follows that
\[
\mathbf{a} = \frac{d^2 \mathbf{x}}{dt^2} = \frac{\ddot{x}_0}{c^2} \mathbf{e}_0 + \frac{\ddot{x}_1}{c^2} \mathbf{e}_1 + \frac{\ddot{x}_2}{c^2} \mathbf{e}_2 + \frac{\ddot{x}_3}{c^2} \mathbf{e}_3.
\]

### 10 points: Problem 1.13
A pulsed radar source is at rest at the origin of the K frame. A meteorite is observed to move with a constant velocity, \( v \), toward the source and is at \( (-\ell, 0, 0) \) at \( t = 0 \). You can assume that the pulses emitted by the radar has zero width in time. A first radar pulse is emitted by the source at \( t = 0 \) and a second pulse at \( t = t_0 < \frac{\ell}{v} \). The meteorite is a perfect reflector of the emitted pulses. Draw the world lines (you need to (a) be precise in your diagram about the angles; (b) differentiate finite line segments from infinite lines and (c) show the coordinates of all intercepts and intersections) in the \( x - ct \) plane for

- the radar source;
- the meteorite;
- the two outgoing pulses; and
- the two reflected pulses.

Repeat the same in the rest frame of the meteorite. From your two diagrams, read off the time difference between the arrival of the reflected pulses at the radar source as measured by the radar and the time difference between the arrival of the two incident pulses at the meteorite as measured by the meteorite. Give a physical explanation of the measurements.

### 10 points: Problem 1.14
Three identical pulse radar sources, A, B and C, are transmitting signals at a frequency, \( f \), in their own rest frame. In the frame where B is at rest, A is moving in the negative \( x \) direction with a velocity \( v \) and C is moving in the positive \( x \) direction with a velocity \( v \). Starting from Eq. (1.40), draw the world lines of the pulses of two of the sources in the rest frame of the third. For clarity, you may want to draw one diagram for each pair (for example, one diagram showing the path of pulses from A in the rest frame of B). Read off the frequency at which one of the sources receives pulses from the other two. All formulas have to be derived starting from Eq. (1.40) and you should show how you read off all six values. Give a physical explanation of your results.

### 1.8 Four vectors – Position, velocity and acceleration

Given the motion of a point particle, \( (ct(\tau), x(\tau), y(\tau), z(\tau)) \), we define the event at \( \tau \) using the four position vector
\[
\mathbf{x}(\tau) = (x(\tau), y(\tau), z(\tau)).
\]

We observe that
\[
d\tau = dt \sqrt{1 - \frac{\mathbf{u} \cdot \mathbf{u}}{c^2}}
\]
where
\[
\mathbf{u} = (u^1, u^2, u^3); \quad u^i(t(\tau)) = \frac{dx^i(\tau)}{d\tau}; \quad i = 1, 2, 3,
\]
is the standard instantaneous velocity of the particle. We define the four velocity vector and the four acceleration vector as
\[
\mathbf{v} = (v^0, v^1, v^2, v^3) = \frac{1}{c} \left( \frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau} \right) = \frac{1}{c^2} \left( \frac{\mathbf{u}}{1 - u \cdot \mathbf{u}/c^2} \right),
\]
and
\[
\mathbf{a} = (a^0, a^1, a^2, a^3) = \frac{1}{c^2} \left( \frac{d^2 x^0}{d\tau^2}, \frac{d^2 x^1}{d\tau^2}, \frac{d^2 x^2}{d\tau^2}, \frac{d^2 x^3}{d\tau^2} \right),
\]
respectively.

Associated with every four position vector with an \textit{up index}, there is a four vector with a \textit{down index}, defined by
\[
x_0 = x^0; \quad x_1 = -x^1; \quad x_2 = -x^2; \quad x_3 = -x^3; \quad x_i = (x_0, x_1, x_2, x_3).
\]

It follows that
\[
c^2 d\tau^2 = \sum_{i=0}^{3} dx_i dx_i,
\]
and
\[
\dot{x}_i \dot{x}_i = 1
\]
where we have assumed that a pair of repeated indices are summed over its range. Noting that
\[ \dddot{x}^i = \frac{\text{d}^2 \dot{x}^i}{\text{d} \tau^2}, \] (1.74)
we see that Eq. (1.73) implies that
\[ \dddot{x}^i \dot{x}_i = \dddot{x}^i \dot{x}_i = 0. \] (1.75)

- **10 points:** Problem 1.15: Using Eq. (1.40) show that \( x^i x_i, \dot{x}^i x_i, \ddot{x}^i x_i, \dddot{x}^i \dot{x}_i \) and \( \dddot{x}^i \dot{x}_i \) are all invariant.

- **10 points:** Problem 1.16: Prove the identity
\[ \dddot{x}^i \dot{x}_i = -\frac{\mathbf{a} \cdot \mathbf{a}}{c^4} (1 - \frac{\mathbf{u} \cdot \mathbf{u}}{c^2})^2 - \frac{(\mathbf{u} \cdot \mathbf{a})^2}{c^6} (1 - \frac{\mathbf{u} \cdot \mathbf{u}}{c^2})^3; \quad \mathbf{a} = \frac{\text{d} \mathbf{u}}{\text{d} \tau}. \] (1.76)

1.9 One dimensional motion under a constant acceleration

One of the first problems in mechanics to understand the role of a force is to study one dimensional motion that produces a constant acceleration. Since the standard instantaneous velocity is a frame dependent quantity, we need to first define what we mean by constant acceleration. To this end we set the strictly non-positive invariant in Eq. (1.76) to be a constant along the path given by
\[ \dddot{x}^i \dot{x}_i = -\frac{w^2}{c^4} \] (1.77)
where \( w \) is a positive constant.

- **5 points:** Problem 1.17: Derive the units of \( w \) and \( \frac{c}{w} \).

Without loss of generality, we can choose the inertial frame such that the motion is along the \( x \) direction. There are, of course, many choices of such inertial frames. Then
\[ \mathbf{u} = (u_x, 0, 0); \quad \mathbf{a} = \left( \frac{\text{d} u_x}{\text{d} \tau}, 0, 0 \right). \] (1.78)
Inserting these into Eq. (1.76) and equating it to \( -\frac{w^2}{c^4} \) using Eq. (1.77), we obtain
\[ w^2 = \left( \frac{\text{d} u_x}{\text{d} \tau} \right)^2 \left( 1 - \frac{u_x^2}{c^2} \right)^3. \] (1.79)
Therefore,
\[ w = \frac{d}{d \tau} \left( \frac{u_x}{\sqrt{1 - \frac{u_x^2}{c^2}}} \right). \] (1.80)
Assuming the initial condition, \( u_x(0) = 0 \), the solution to the velocity as a function of time is
\[ u_x(t) = \frac{wt}{\sqrt{1 + \frac{w^2 t^2}{c^2}}} = \begin{cases} \frac{wt}{c} - \frac{1}{2} \frac{w^3 t^3}{c^3} + \cdots & t \ll \frac{c}{w} \\ \frac{c}{2} \frac{w^3 t^3}{c^3} + \cdots & t \gg \frac{c}{w} \end{cases}. \] (1.81)

- **10 points:** Problem 1.18: Derive an expression for the position \( x(t) \) as a function of \( t \) and obtain expressions for \( t \ll \frac{c}{w} \) and \( t \gg \frac{c}{w} \). Discuss the physical meaning of your results.

- **10 points:** Problem 1.19: Derive an expression for the proper time \( \tau(t) \) as a function of \( t \) and obtain expressions for \( t \ll \frac{c}{w} \) and \( t \gg \frac{c}{w} \). Discuss the physical meaning of your results.

- **10 points:** Problem 1.20: Derive expressions for the four position vector, four velocity vector and four acceleration vector in terms of \( w, \tau \) and \( c \). Discuss your results using your knowledge of Lorentz transformations.
Chapter 2

Physics of charged particles

Classical physics will be dictated by the principle of least action: *Equations of motion are obtained by minimizing the invariant distance between the initial and final points.* We will only concern ourselves with the interaction of point particles that have masses and charges. We will start with the discussion of a free particle. We will then introduce the interaction between the charge of this particle and an external electromagnetic field. Finally, we will discuss the dynamics of the electromagnetic field itself.

2.1 Free point particle

The action is proportional to the invariant distance given by

\[
S = -\alpha c \int_i^f \mathrm{d}r.
\]

(2.1)

The negative sign is introduced to define the action as an invariant distance as opposed to an invariant time. Using Eq. (1.67), we have

\[
S = \int_i^f L \mathrm{d}t; \quad \text{and} \quad L = -\alpha c \sqrt{1 - \frac{u \cdot u}{c^2}},
\]

(2.2)

is the Lagrangian. Expanding the Lagrangian in powers of \( \frac{u \cdot u}{c^2} \), we obtain

\[
L = -\alpha c + \frac{\alpha}{2c} u \cdot u + \cdots.
\]

(2.3)

The first term is a constant and does not affect the equations of motion. The second term becomes the non-relativistic kinetic energy of a particle of mass \( m \) if we set \( \alpha = mc \). Therefore, the action for a free point particle of mass \( m \) is

\[
S = -mc^2 \int_i^f \mathrm{d}r,
\]

(2.4)

and the Lagrangian is

\[
L = -mc^2 \sqrt{1 - \frac{u \cdot u}{c^2}}.
\]

(2.5)

As expected, the Lagrangian of the free particle depends only on its velocity and not on its position. The equations of motion is simply the conservation of momentum,

\[
\frac{dp}{dt} = 0; \quad p = \frac{\partial L}{\partial u} = \frac{mu}{\sqrt{1 - \frac{u \cdot u}{c^2}}}.
\]

(2.6)

- **5 points**: Problem 2.1: Derive the two terms that contribute to \( \frac{dp}{dt} \) and discuss their physical significance.

- **10 points**: Problem 2.2: Prove that the conservation of momentum is indeed a minimum by computing the second derivatives of the Lagrangian with respect to the velocities.
The Hamiltonian of the free point particle is obtained from the Legendre transformation,

\[ \mathcal{H}(p) = p \cdot u - L = \frac{mc^2}{\sqrt{1 - \frac{u \cdot u}{c^2}}} \]  

(2.7)

where we have used the Lagrangian in Eq. (2.5) and the momentum in Eq. (2.6) to obtain the last equality. We are not done since we need to invert Eq. (2.6) to write \( H(p) \).

**• 10 points: Problem 2.3:** Show that

\[ \frac{dH}{dt} = 0, \]  

(2.8)

using Eq. (2.7) and Eq. (2.6).

Since the Hamiltonian is a constant of the motion and since the point particle is free, we refer to \( E_{\text{kin}} = \frac{mc^2}{\sqrt{1 - \frac{u \cdot u}{c^2}}} \)

(2.9)

as the relativistic kinetic energy of the point particle with mass \( m \). Expanding in powers of \( \frac{u \cdot u}{c^2} \), we obtain

\[ E_{\text{kin}} = mc^2 + \frac{1}{2} m (u \cdot u) + \cdots. \]  

(2.10)

The first term is the rest energy of the particle and the second term is its non-relativistic kinetic energy.

In order to obtain the energy (Hamiltonian) in terms of the momentum, we observe using Eq. (2.7) and Eq. (2.9) that

\[ \mathcal{H}^2 - (p \cdot p) c^2 = m^2 c^4 \quad \Rightarrow \quad \mathcal{H} = c \sqrt{p \cdot p + m^2 c^2} = E_{\text{kin}}. \]  

(2.11)

As expected, the Hamiltonian only depends on the momentum of the particle and not on its coordinates.

**• 10 points: Problem 2.4:** Derive the equations of motion starting from the Hamiltonian; namely, use the Hamilton’s equations,

\[ \frac{dp}{dt} = - \frac{\partial \mathcal{H}}{\partial x}; \quad \frac{dx}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \]  

(2.12)

to derive the two sets of equations of motion for the free point particle. Discuss these two sets of equations in the context of what we have seen before in this section.

We note that

\[ p^i = \left( \frac{E_{\text{kin}}}{c}, p \right) = \frac{mc}{\sqrt{1 - \frac{u \cdot u}{c^2}}} \left( 1, \frac{u}{c} \right) = mc \dot{x}^i, \]  

(2.13)

where we have used Eq. (2.9) and Eq. (2.6) to obtain the second equality and the expression for the four-velocity in Eq. (1.69) to obtain the last equality. Therefore, \( p^i \) is a four-vector and its components transform under a Lorentz transformation according to Eq. (1.40), namely,

\[ \begin{pmatrix} \xi'_{\text{kin}} \\ p'_{z} \\ p'_{y} \\ p'_{x} \end{pmatrix} = \begin{pmatrix} \gamma & \beta \gamma & 0 & 0 \\ \beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_{\text{kin}} \\ p_{z} \\ p_{y} \\ p_{x} \end{pmatrix}. \]  

(2.14)

Noting that

\[ p_i = \left( \frac{E_{\text{kin}}}{c}, -p \right) = mc \dot{x}_i, \]  

(2.15)

we see that Eq. (2.11) is nothing but

\[ p_i p^i = m^2 c^4; \quad \text{since} \quad \dot{x}_i \dot{x}^i = 1. \]  

(2.16)

**• 5 points: Problem 2.5:** Show from what we have learned before in this section that

\[ \frac{dp^i}{c d\tau} = 0 \quad \Rightarrow \quad \dot{x}^i = 0. \]  

(2.17)
2.2 Interaction of a charged particle with an external electromagnetic field

Let us assume that our point particle with mass, \( m \), also has a charge, \( e \). The charges enables an interaction with an external electromagnetic field. This external field is itself due to other charged particle but we will assume for the moment that it is given. In other words, we are assuming that our charged particle is a test particle: The external field can affect its motion but it cannot affect the external field.

2.2.1 Action, Lagrangian and Hamiltonian

We need to modify our action. We still assume that action describes a path but the external field changes the distance between two points. Let us consider a four-vector field, \( A_i(x^0, x^1, x^2, x^3) \), which changes the invariant distance from

\[
-mc^2d\tau \rightarrow -mc^2d\tau - \frac{e}{c}A_idx^i. \tag{2.18}
\]

The change is present only if the vector field is non-zero at the location and the change is linear in this field and proportional to the charge, \( e \). Our aim is to derive the equations of motion for the point particle starting with the modified invariant distance.

We write

\[
A^i = (\phi, A); \quad A_i = (\phi, -A) \tag{2.19}
\]

where \( \phi \) and \( A \) are the already familiar scalar potential field and vector potential field. Then,

\[
S = \int^f \left( -mc^2d\tau - \frac{e}{c}A_idx^i \right) = \int^f \left( -mc^2d\tau - e\phi dt + \frac{e}{c}A \cdot d\mathbf{x} \right). \tag{2.20}
\]

The Lagrangian for the charged particle interacting with an external electromagnetic field is

\[
S = \int^f Ldt \quad L(t, \mathbf{x}, \mathbf{u}) = -mc^2\sqrt{1 - \frac{\mathbf{u} \cdot \mathbf{u}}{c^2}} + \frac{e}{c}A \cdot \mathbf{u} - e\phi. \tag{2.21}
\]

The dependence of the Lagrangian on the space and time coordinates arises from the dependence of the scalar and vector potential fields on space and time. The momenta conjugate to \( \mathbf{x} \) are

\[
\pi = \frac{\partial L}{\partial \mathbf{u}} = \mathbf{p} + \frac{e}{c}\mathbf{A}. \tag{2.22}
\]

The Hamiltonian given by the Legendre transformation is

\[
\mathcal{H} = \mathbf{u} \cdot \pi - L = \mathbf{u} \cdot \mathbf{p} + mc^2\sqrt{1 - \frac{\mathbf{u} \cdot \mathbf{u}}{c^2}} + e\phi. \tag{2.23}
\]

The first two terms together is the Hamiltonian for the free particle and therefore

\[
\mathcal{H} = \mathcal{E}_{\text{kin}} + e\phi. \tag{2.24}
\]

The four momentum vector is

\[
p^i = \left( \frac{\mathcal{E}_{\text{kin}}}{c}, \mathbf{p} \right) = \left( \frac{\mathcal{H} - e\phi}{c}, \pi - \frac{e}{c}\mathbf{A} \right). \tag{2.25}
\]

From

\[
p_ip^i = m^2c^4, \tag{2.26}
\]

we have

\[
\left( \frac{\mathcal{H} - e\phi}{c} \right)^2 - \left( \pi - \frac{e}{c}\mathbf{A} \right) \cdot \left( \pi - \frac{e}{c}\mathbf{A} \right) = m^2c^2. \tag{2.27}
\]

Therefore, the Hamiltonian is

\[
\mathcal{H}(t, \pi, \mathbf{x}) = \sqrt{m^2c^4 + c^2\left( \pi - \frac{e}{c}\mathbf{A} \right) \cdot \left( \pi - \frac{e}{c}\mathbf{A} \right)} + e\phi. \tag{2.28}
\]
2.2.2 Equations of motion

The equations of motion for the charged particle are given by the Euler-Lagrange equations,

\[
\frac{d}{dt} \pi = \frac{\partial L}{\partial x} = \frac{e}{c} (\nabla A) \cdot u - e \nabla \phi; \quad \left[(\nabla A) \cdot u\right]_i = (\partial^i A^j) u^j; \quad \partial^j = \frac{\partial}{\partial x^j},
\]

and repeated indices are summed from 1 to 3. Using the expression for the conjugate momenta in Eq. (2.22), we arrive at

\[
\frac{dp}{dt} = -\frac{e}{c} \frac{dA}{dt} + \frac{e}{c} (\nabla A) \cdot u - \frac{e}{c} \left(\nabla \phi \right) = -\frac{e}{c} \nabla \phi - \frac{e}{c} \frac{\partial A}{\partial t} - \frac{e}{c} (u \cdot \nabla) A + \frac{e}{c} (\nabla A) \cdot u.
\]

Noting that (using summation over repeated indices)

\[
\left[(\nabla A) \cdot u - (u \cdot \nabla) A\right]_i = (\partial^i A^j) u^j - w^i (\partial^j A^j) = (\delta^i_1 \delta^j_m - \delta^i_m \delta^j_1) u^j (\partial^j A^m) = \epsilon^{ijk} u^j (\partial^k A^m),
\]

we arrive at the equations of motion driven by the Lorentz force

\[
\frac{dp}{dt} = eE + \frac{e}{c} u \times H,
\]

where

\[
E = -\nabla \phi - \frac{1}{c} \frac{\partial A}{\partial t}; \quad H = \nabla \times A
\]

are the external electric and magnetic fields respectively. Note that we have \(\frac{dp}{dt}\) on the left hand side and not the non-relativistic \(m \frac{du}{dt}\).

Since both

\[
\frac{(E_{\text{kin}})^2}{c^2} - p \cdot p = m^2 c^2; \quad E_{\text{kin}} u = c^2 p
\]

remain true for the charged particle interacting with an external electromagnetic field, we take the time derivative of the first equation and use the second equation to arrive at

\[
E_{\text{kin}} \frac{dE_{\text{kin}}}{dt} = c^2 p \cdot \frac{dp}{dt} \quad \Rightarrow \quad \frac{dE_{\text{kin}}}{dt} = u \cdot \frac{dp}{dt}.
\]

Upon using the equations of motion in Eq. (2.32) we obtain

\[
\frac{dE_{\text{kin}}}{dt} = cu \cdot E,
\]

which states that a change in the kinetic energy of the particle can only occur due to the electric field.

**10 points: Problem 2.6:** Show that

\[
m \frac{du}{dt} = e \sqrt{1 - \frac{u \cdot u}{c^2}} \left[\frac{1}{c} u \times H - \frac{1}{c^2} (u \cdot E) u\right].
\]

Expand the right-hand side in powers of \(\frac{u}{c}\) to second order and discuss the physics of each one of the terms.

2.2.3 Electromagnetic field tensor

It will be useful to re-derive the equations of motion in a Lorentz covariant form. That is to say, we will only use four vectors and related quantities and use up and down indices. We will also use the summation over repeated indices convention where the sum will run from 0 to 3. To this end we write

\[
e^2 d\tau^2 = dx_j dx^j,
\]

and our action becomes

\[
S = - \int_1^f \left( mc \sqrt{dx_j dx^j} + \frac{e}{c} A_j dx^j \right).
\]
2.2. INTERACTION OF A CHARGED PARTICLE WITH AN EXTERNAL ELECTROMAGNETIC FIELD

We consider the variation of $S$ when we change $x^j$ to $(x^j + \delta x^j)$ independently at every point along the path subject to the condition that $\delta x^j$ is zero at the initial and final points. Then

$$0 = \delta S = - \int_{j}^{i} \left[ mc \frac{dx_j}{c d\tau} \delta x^j + \frac{e}{c} A_j \delta x^j + \frac{e}{c} A_j dx^j \right] = - \int_{i}^{j} \left[ mc d \left( \frac{dx_j}{c d\tau} \right) \delta x^j + \frac{e}{c} dA_j \delta x^j - \frac{e}{c} \delta A_j dx^j \right] = \int_{i}^{j} \left[ mc \frac{d\dot{x}_j}{c d\tau} \delta x^j + \frac{e}{c} \partial A_j \frac{dx^k}{c dx^j} \delta x^j - \frac{e}{c} \partial A_k \frac{dx^j}{c dx^k} \right] c d\tau \delta x^j \quad (2.40)$$

We have integrated by parts in obtaining the third line from the second and used the fact that $\delta x^j$ is zero at the initial and final points. We have used the four velocity vector defined in Eq. (1.69) in going from the third to fourth line. We have also used the fact that the variation in the external four vector field, $A_j$, is due to the change in the path. We have multiplied and divided by $c d\tau$ in going from the fourth to the fifth line and we have also interchanged the summation indices $(j, k)$ in the last term so that we can extract a common factor of $\delta x^j$ from all terms. Since the variation, $\delta x^j$, is independent at every point along the path, an extremum is obtained if and only if

$$mc \ddot{x}_j = \frac{e}{c} \left( \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} \right) \dot{x}^k, \quad (2.41)$$

where we have used the four velocity and four acceleration vectors defined in Eq. (1.69) and Eq. (1.70). The equations of motion in a Lorentz covariant form is

$$mc \ddot{x}_j = - \frac{e}{c} F_{jk} \dot{x}^k, \quad (2.42)$$

where

$$F_{jk} = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}, \quad (2.43)$$

is the anti-symmetric electromagnetic field tensor with two down indices. Using $A_j = (\phi, -\mathbf{A})$ and $x^i = (ct, \mathbf{x})$, it follows that the electromagnetic field tensor with two down indices in matrix notation is

$$F_{jk} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix}. \quad (2.44)$$

- **5 points: Problem 2.7:** Show that the identity $\dot{x}^j \ddot{x}_j = 0$ derived in Section 1.8 is consistent with Eq. (2.42).

- **10 points: Problem 2.8:** Use expressions derived before in this chapter to re-write Eq. (2.42) as

$$\frac{dp_j}{c d\tau} = \frac{e}{c} F_{jk} \dot{x}^k. \quad (2.45)$$

Write down the four equations $(j = 0, 1, 2, 3)$ explicitly in terms of $E_{\text{kin}}, \mathbf{p}, \mathbf{E}$ and $\mathbf{H}$ and identify them with previously derived equations.

- **10 points: Problem 2.9:** The four vector field $A'$ transforms according to Eq. (1.40) under a transformation that relates two inertial frames moving with respect to each other in the $x$ direction at a constant speed, $v$. Since the electric and magnetic fields are part of an electromagnetic field tensor with two down indices, their transformation properties have to derived from the transformation properties of the electromagnetic field tensor. Perform this derivation and show that

$$E'_x = E_x; \quad E'_y = \gamma(E_y + \beta H_z); \quad E'_z = \gamma(E_z - \beta H_y); \quad H'_x = H_x; \quad H'_y = \gamma(H_y - \beta E_z); \quad H'_z = \gamma(H_z + \beta E_y). \quad (2.46)$$
Consider the fully antisymmetric tensor with four up indices, \( \epsilon^{jklm} \), such that \( \epsilon^{0123} = 1 \). Since

\[
\det \Lambda = \epsilon^{jklm} \Lambda_j \Lambda_{k} \Lambda_{l} \Lambda_{m} = 1,
\]

from Eq. (1.40), it follows that \( \epsilon^{jklm} \) is an invariant tensor – its components remain the same in all inertial frames. Given the electromagnetic field tensor with two down indices, we can form a dual to the electromagnetic field tensor with two up indices by

\[
\tilde{F}^{jk} = \frac{1}{2} \epsilon^{jklm} F_{lm},
\]

and is also anti-symmetric. Explicitly,

\[
\tilde{F}^{jk} = \begin{pmatrix}
0 & -H_x & -H_y & -H_z \\
H_x & 0 & E_z & -E_y \\
H_y & -E_z & 0 & E_x \\
H_z & E_y & E_x & 0
\end{pmatrix},
\]

and we see that the dual simply switches \( E \leftrightarrow -H \).

Since the Lorentz transformation, by definition, leaves the distance between two four vectors invariant, it follows that the metric tensor, \( g^{jk} \), defined by

\[
g^{jk} = \begin{cases}
0; \quad & j \neq k; \\
g_{00} = 1; \\
g_{11} = g_{22} = g_{33} = -1;
\end{cases}
\]

is an invariant tensor. Furthermore,

\[
g^{jk} = g_{jk}
\]

is also an invariant tensor. It is also clear that we can raise or lower an index using the metric tensor:

\[
f^{j} = g_{jk}f^{k}; \quad f^{j} = g^{jk}f_{k}
\]

for any four vector \( f_{j} \). Therefore, we have

\[
F^{jk} = g^{jl}g^{km} F_{lm}; \quad \tilde{F}^{jk} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & H_z & H_y \\
E_y & H_z & 0 & -H_x \\
E_z & -H_y & H_x & 0
\end{pmatrix},
\]

and

\[
\tilde{F}^{jk} = g_{jl}g_{km} F_{lm}; \quad \tilde{F}^{jk} = \begin{pmatrix}
0 & H_x & H_y & H_z \\
-H_x & 0 & E_z & -E_y \\
-H_y & -E_z & 0 & E_x \\
-H_z & E_y & E_x & 0
\end{pmatrix}
\]

We can form an invariant using Eq. (2.53) and Eq. (2.44) or using Eq. (2.49) and Eq. (2.54), namely,

\[
-\frac{1}{2} F^{jk} F_{jk} = \frac{1}{2} \tilde{F}^{jk} \tilde{F}_{jk} = E \cdot E - H \cdot H,
\]

and we can form another invariant using Eq. (2.53) and Eq. (2.54), namely,

\[
-\frac{1}{4} \tilde{F}^{jk} F_{jk} = E \cdot H.
\]

### 2.3 Gauge invariance

Since the interaction term of the action in Eq. (2.20) is of the form

\[
S_{\text{int}} = -\frac{e}{c} \int f A_i dx^i,
\]

it follows that a change of the form

\[
A_i = A'_i + \frac{\partial \chi}{\partial x^i},
\]

is a gauge transformation.
for an arbitrary scalar function, \( \chi \), only changes the action by a constant. To see this, note that
\[
S_{\text{int}}' = \int f' A' dx^i = \frac{e}{c} \int f' A dx^i + \frac{e}{c} \int f' d\chi = S_{\text{int}} + \frac{e}{c} (\chi f - \chi i).
\]
As such a change of the form
\[
A' = A + \nabla \chi; \quad \phi' = \phi - \frac{1}{c} \frac{\partial \chi}{\partial t},
\]
will not change the equations of motion and therefore correspond to the same physical electromagnetic field. To further emphasize this point, note that the electromagnetic field tensor does not change under Eq. (2.58):
\[
F'_{jk} = \frac{\partial A'_{k}}{\partial x^j} - \frac{\partial A'_{j}}{\partial x^k} - \frac{\partial^2 \chi}{\partial x^j \partial x^k} + \frac{\partial^2 \chi}{\partial x^k \partial x^j} = F_{jk}.
\]
Therefore, the physical electric and magnetic fields do not change under Eq. (2.58). The transformation in Eq. (2.58) that groups an infinite set of vector fields into one physical electromagnetic field is referred to as a gauge transformation. We will use this transformation later on to our advantage and choose a nice representative vector field.

### 2.4 Classification of constant and uniform electromagnetic fields

Let us consider constant and uniform electromagnetic fields, namely fields that do not vary in space or time. Noting that only \( E \) and \( H \) have physical content and noting that we have the freedom to choose an inertial reference frame thereby changing our electric and magnetic field, we see that we can classify constant and uniform electric and magnetic fields into the following cases using Eq. (2.55) and Eq. (2.56).

Given a constant and uniform electric and magnetic field, we can choose the spatial coordinates in our inertial reference frame such that \( j \) is along \( E \), namely,
\[
E = E_j; \quad E > 0
\]
and \( H \) has components only in the \( j \) and \( k \) directions, namely,
\[
H = H_j j + H_k k; \quad H_z \geq 0.
\]

1. \( E \cdot H = 0 \Rightarrow H_y = 0 \): In this case we can write
\[
E = E_j; \quad H = H_k; \quad E > 0; \quad H > 0.
\]
Under a Lorentz transformation where one inertial frame is moving with respect to the other with a velocity \( v_i \), the electric and magnetic fields transform according to Eq. (2.46) which reduces to
\[
E' = E'_j; \quad E' = \gamma (E + \beta H) \quad \text{and} \quad H' = H'_k; \quad H' = \gamma (H + \beta E).
\]
We have to consider only three cases:

(a) \( E = H \): For this case,
\[
E' = H' = \gamma (1 + \beta) E = \sqrt{1 + \beta \frac{E}{1 - \beta}} E.
\]
Therefore the case of

Case D : \( E = E_j \) and \( H = E_k \)

transforms into itself with a new value for \( E \).

(b) \( E > H \): For this case we can choose
\[
\beta = \frac{v}{c} = - \frac{H}{E}; \quad \gamma = \frac{E}{\sqrt{E^2 - H^2}}
\]
and we have
\[
E' = \sqrt{E^2 - H^2}; \quad H' = 0.
\]
This case reduces to a problem with just an electric field, namely,

Case A : \( E = E_j \) and \( H = 0 \).
(c) $E < H$: For this case we can choose

$$\beta = \frac{v}{c} = \frac{E}{H}, \quad \gamma = \frac{H}{\sqrt{H^2 - E^2}}$$

and we have

$$E' = 0; \quad H' = \sqrt{H^2 - E^2}.$$  

This case reduces to a problem with just a magnetic field, namely,

**Case B**: $E = 0$; and $H = Hk$.

2. $E \cdot H \neq 0$: Under a Lorentz transformation where one inertial frame is moving with respect to the other with a velocity $v$, the electric and magnetic fields transform according to Eq. (2.46) which reduces to

$$E' = \gamma [(E + \beta H_z)j - \beta H_yk]; \quad H' = \gamma [H_yj + (H_z + \beta E)k].$$  

**10 points**: **Problem 2.10**: Prove that the two invariances in Eq. (2.55) and Eq. (2.56) indeed remain invariant.

Assuming $E \times H = EH_z \neq 0$,

we can use the freedom in choosing $\beta$ and require that

$$\mathbf{E}' \times \mathbf{H}' = 0,$$

making $\mathbf{E}'$ parallel to $\mathbf{H}'$. Upon inserting Eq. (2.74) into Eq. (2.76), we arrive at

$$\gamma^2 \left[ (E + \beta H_z)(H_z + \beta E) + \beta H_y^2 \right] i = 0 \Rightarrow \beta^2 + \frac{E^2 + H_y^2 + H_z^2}{EH_z} \beta + 1 = 0.$$  

**10 points**: **Problem 2.11**: Prove that

$$\frac{E^2 + H_y^2 + H_z^2}{EH_z} \geq 2.$$  

Due to the above proof we can conclude that this case can be reduced to a problem with parallel electric and magnetic field, namely,

**Case C**: $E = Ej$; and $H = Hj$.

2.4.1 **Case A**: **Solution to $E = Ej$ and $H = 0$**

The equations of motion for this case given by Eq. (2.32) reduce to

$$\frac{dp_x}{dt} = 0; \quad \frac{dp_y}{dt} = eE; \quad \frac{dp_z}{dt} = 0; \Rightarrow p_x = p_{x0}; \quad p_y = p_{y0} + eEt; \quad p_z = p_{z0}.$$  

We are free to choose our definition for $t = 0$ and we will use this to set $p_{x0} = 0$. The electric field defines the $j$ direction and we are free to choose our $i$ and $k$ direction. We will do this by assuming that the momentum at $t = 0$ is in the $i$ direction. We will set, $p_{x0} = p_0$, the magnitude of the momentum at $t = 0$, and $p_{z0} = 0$. The momentum as a function of time is, therefore, given by

$$p = p_0 i + eEtj.$$  

Using Eq. (2.34), the kinetic energy of the particle is given by

$$\mathcal{E}_{\text{kin}} = \sqrt{m^2c^4 + c^2p_0^2 + (ceEt)^2} = \sqrt{\mathcal{E}_0^2 + (ceEt)^2},$$  

where

$$\mathcal{E}_0 = \sqrt{m^2c^4 + c^2p_0^2}$$  

is the initial kinetic energy of the particle. It also follows from Eq. (2.34) that

$$\frac{dx}{dt} = u = \frac{c^2p}{\mathcal{E}_{\text{kin}}} = \frac{c^2(p_0i + eEtlj)}{\sqrt{\mathcal{E}_0^2 + (ceEt)^2}}.$$  

\[2.81\]
2.4. CLASSIFICATION OF CONSTANT AND UNIFORM ELECTROMAGNETIC FIELDS

- 10 points: Problem 2.12: Using our freedom to set the origin of the coordinate system as the location of the particle at \( t = 0 \), show that

\[
x = \left[ \frac{p_0 c}{e E} \sinh^{-1} \left( \frac{c E t}{E_0} \right) \right] i + \left[ \frac{1}{c E} \left( \sqrt{E_0^2 + (c E t)^2} - E_0 \right) \right] j.
\]

(2.85)

Use the first three non-zero terms in the Taylor expansion and discuss the short time \( t << \frac{E_0}{c E} \) and long term \( t >> \frac{E_0}{c E} \) behavior of both \( x(t) \) and \( u(t) \). Derive an expression for the path of the particle in the form of \( y(x) \) and discuss its small \( x \) and large \( x \) behavior using the Taylor expansion to two non-zero orders.

2.4.2 Case B: Solution to \( E = 0 \) and \( H = H k \)

For this case we use Eq. (2.32) and Eq. (2.36) and they reduce to

\[
\begin{align*}
\frac{dp_x}{dt} &= \frac{e}{c} u_y H; & \frac{dp_y}{dt} &= -\frac{e}{c} u_x H; & \frac{dp_z}{dt} &= 0; & \frac{dE_{\text{kin}}}{dt} &= 0.
\end{align*}
\]

(2.86)

We use Eq. (2.34) to write \( p \) in terms of \( u \) and also use the fact that \( E_{\text{kin}} \) is a constant in time to obtain

\[
\begin{align*}
\frac{du_x}{dt} &= \omega u_y; & \frac{du_y}{dt} &= -\omega u_x; & \frac{du_z}{dt} &= 0;
\end{align*}
\]

(2.87)

where

\[
\omega = \frac{ec H}{E_{\text{kin}}}.
\]

(2.88)

is a constant and defines the angular frequency of the circular motion in the plane perpendicular to the magnetic field. Using our freedom to set \( t = 0 \), we define \( t = 0 \) as the time when the velocity perpendicular to the magnetic field is pointing in the \( j \) direction. The solutions to the velocities are given by

\[
\frac{dx}{dt} = u_\perp \left[ \sin(\omega t) i + \cos(\omega t) j \right] + u_\parallel k,
\]

(2.89)

where both \( u_\perp \) and \( u_\parallel \) are constants.

- 5 points: Problem 2.13: Derive the expression that relates \( E_{\text{kin}} \) in terms of \( u_\perp \) and \( u_\parallel \). Hence obtain an expression for \( \omega \) in terms of \( m, e, c, H \) and \( u = \sqrt{u_\perp^2 + u_\parallel^2} \). Discuss the physics of the behavior of \( \omega \) for \( u << c \) and also as \( u \) approaches \( c \).

Using our freedom to set the origin of the coordinate system as the location of the particle at \( t = 0 \), we arrive at

\[
x(t) = \frac{u_\perp}{\omega} \left[ -\cos(\omega t) i + \sin(\omega t) j \right] + u_\parallel t k.
\]

(2.90)

- 5 points: Problem 2.14: Discuss the motion of the particle for \( u << c \) and also as \( u \) approaches \( c \).

2.4.3 Case C: Solution to \( E = E j \) and \( H = H j \)

For this case, Eq. (2.32) reduce to

\[
\begin{align*}
\frac{dp_x}{dt} &= -\frac{e}{c} u_z H; & \frac{dp_y}{dt} &= e E; & \frac{dp_z}{dt} &= \frac{e}{c} u_x H.
\end{align*}
\]

(2.91)

Defining \( t = 0 \) as the point where \( p_y = 0 \) we arrive at

\[
p_y = e E t.
\]

(2.92)

Furthermore, it follows from the first and third equations in Eq. (2.91) that

\[
\frac{d(p_x^2 + p_z^2)}{dt} = 0.
\]

(2.93)

From Eq. (2.34), we obtain

\[
E_{\text{kin}}^2 = m^2 c^4 + (p_x^2 + p_z^2) c^2 + (c E t)^2 = E_0^2 + (c E t)^2.
\]

(2.94)
where $E_0$ is the energy at $t = 0$. Using Eq. (2.102), we can write $u$ in terms of $p$ to obtain

$$\frac{dp_x}{dt} = \frac{c e H}{E_{\text{kin}}} p_z; \quad \frac{dp_z}{dt} = \frac{c e H}{E_{\text{kin}}} p_x. \quad (2.95)$$

We will introduce complex variables to solve this problem by defining

$$p_x + ip_z = p_\perp e^{i\phi(t)}; \quad p_\perp^2 = p_z^2 = p_x^2.$$

It follows from Eq. (2.93) that $p_\perp$ is a constant. Therefore,

$$ip_\perp e^{i\phi} \frac{d\phi}{dt} = \frac{d(p_x + ip_z)}{dt} = \frac{c e H}{E_{\text{kin}}} (-p_z + ip_x) = \frac{c e H}{E_{\text{kin}}} (p_x + ip_z) = \frac{c e H}{E_{\text{kin}}} p_\perp e^{i\phi}. \quad (2.97)$$

Using Eq. (2.94), the above equation reduces to

$$\frac{d\phi}{dt} = \frac{c e H}{\sqrt{E_0^2 + (c e H)^2}} \Rightarrow \phi = \frac{H}{E} \sinh^{-1} \left( \frac{c e H}{E_0} \right). \quad (2.98)$$

and we used our degree of freedom to choose the $i$ and $k$ directions in setting $\phi = 0$ at $t = 0$. Next, we use Eq. (2.34) to write $p_x$ and $p_z$ in terms of $u_x$ and $u_z$ to obtain

$$p_\perp e^{i\phi} = p_x + ip_y = \frac{E_{\text{kin}}}{c^2} (u_x + iu_y) = \frac{E_{\text{kin}}}{c^2} \frac{d(x + iy)}{dt} = \frac{E_{\text{kin}}}{c^2} \frac{d\phi}{dt} \frac{d(x + iy)}{d\phi}. \quad (2.99)$$

Using Eq. (2.98), we obtain

$$p_\perp e^{i\phi} = \frac{e H}{c} \frac{d(x + iy)}{d\phi} \Rightarrow x + iy = -i \frac{c p_\perp}{e H} e^{i\phi}. \quad (2.100)$$

**10 points: Problem 2.15:** Discuss the motion of the particle in detail with emphasis on the clarity of the presentation from the physics viewpoint.

### 2.4.4 Case D: Solution to $E = Ej$ and $H = Ek$

For this case, Eq. (2.32) and Eq. (2.36) reduce to

$$\frac{dp_x}{dt} = \frac{e}{c} u_y E; \quad \frac{dp_y}{dt} = e E - \frac{e}{c} u_x E; \quad \frac{dp_z}{dt} = 0; \quad \frac{dE_{\text{kin}}}{dt} = e E u_y. \quad (2.101)$$

We use the first and last equations in Eq. (2.101) to arrive at

$$\frac{d\phi}{dt} = \frac{E_{\text{kin}}}{c} \frac{d\phi}{dt} \Rightarrow E_{\text{kin}} - cp_x = \alpha, \quad (2.102)$$

where $\alpha$ is a constant of motion. Furthermore, it follows from the third equation in Eq. (2.101) that $p_z$ is also a constant of motion. Using Eq. (2.34), we have

$$c^2 p_y^2 + \epsilon^2 = E_{\text{kin}}^2 - c^2 p_x^2 = \alpha (E_{\text{kin}} + cp_x); \quad \epsilon^2 = c^2 p_x^2 + m^2 c^4. \quad (2.103)$$

where we have used Eq. (2.102) and $\epsilon$ is a constant. From Eq. (2.102) and Eq. (2.103), we obtain

$$E_{\text{kin}} = \frac{\alpha}{2} + \frac{c^2 p_y^2 + \epsilon^2}{2\alpha}; \quad p_x = \alpha 2c + \frac{c^2 p_y^2 + \epsilon^2}{2\alpha c}. \quad (2.104)$$

We rewrite the second equation in Eq. (2.101) as

$$E_{\text{kin}} \frac{dp_y}{dt} = e E \left( E_{\text{kin}} - \frac{E_{\text{kin}} u_x}{c} \right) = e E (E_{\text{kin}} - cp_x) = e E \alpha, \quad (2.105)$$

and we have used Eq. (2.34) in going from the second to third equality and we have used Eq. (2.102) in arriving at the last equality. Inserting the expression for $E_{\text{kin}}$ in terms of $p_y$ from Eq. (2.104), we obtain the following first order differential equation for $p_y$:

$$\left[ \frac{\alpha}{2} + \frac{c^2 p_y^2 + \epsilon^2}{2\alpha} \right] dp_y = e E \alpha dt. \quad (2.106)$$
Integrating this equation results in

\[ 2eEt = \left(1 + \frac{e^2}{\alpha^2}\right) p_y + \frac{e^2}{3\alpha^2} p_y^3, \quad (2.107) \]

where we have assumed that \( p_y = 0 \) at \( t = 0 \). In order to derive the time evolution of \( x \), we use Eq. (2.34) and Eq. (2.104) and write

\[ \frac{c^2 p}{\epsilon \text{kin}} = \frac{dx}{dt} = \frac{dp_y}{dt} \frac{dx}{dp_y} \quad \Rightarrow \quad \frac{dx}{dp_y} = \frac{c^2 p}{eE\alpha}. \quad (2.108) \]

Using Eq. (2.104) and using the fact that \( p_z \) is a constant, the above equations become

\[ \frac{dx}{dp_y} = \frac{c}{2eE} \left( -1 + \frac{e^2}{\alpha^2} \right) p_y + \frac{c^3}{6eE\alpha^2} p_y^3; \quad \frac{dy}{dp_y} = \frac{c^2}{eE\alpha} p_y^2; \quad \frac{dz}{dp_y} = \frac{c^2 p_z}{cE\alpha} \quad (2.109) \]

and their solutions assuming that the particle is at the origin at \( t = 0 (p_y = 0) \) are

\[ x = \frac{c}{2eE} \left( -1 + \frac{e^2}{\alpha^2} \right) p_y + \frac{c^3}{6eE\alpha^2} p_y^3; \quad y = \frac{c^2}{2eE\alpha} p_y^2; \quad z = \frac{c^2 p_z}{cE\alpha} p_y, \quad (2.110) \]

- 10 points: Problem 2.16: Prove that the time \( t \) uniquely determines a value of \( p_y \). Use any plotting routine of your choice and plot \( x \) as a function of \( t \). Derive expressions for \( u \) and plot these as a function of \( t \). Discuss the motion of the particle in detail with emphasis on the clarity of the presentation from the physics viewpoint.

2.4.5 The grand finale

- 20 points: Problem 2.17: Consider the general case where \( E = E_1 \hat{i} + E_2 \hat{j} + E_3 \hat{k} \) and \( H = H_1 \hat{i} + H_2 \hat{j} + H_3 \hat{k} \). Derive explicit expressions for the motion of the particle, \( x(t) \), in this reference frame.

2.5 One physical example of a constant non-uniform electromagnetic field

Once we relax the condition that the electromagnetic field is non-uniform in space, we can think of a plethora of examples even if the fields are constant in time. In this section we will focus on an example of paramount importance – the case of a central potential of the form

\[ \phi(x, t) = \frac{k}{r}; \quad A(x, t) = 0; \quad r = \sqrt{x^2 + y^2 + z^2}, \quad (2.111) \]

where \( k \) can be either positive or negative. Using the definitions of electric and magnetic field in Eq. (2.33), we find

\[ E = -k \nabla \frac{1}{r} = \frac{k}{r^2} \hat{r}; \quad H = 0. \quad (2.112) \]

The equations of motion, Eq. (2.32) and Eq. (2.36), reduce to

\[ \frac{dp}{dt} = \frac{\alpha}{r^2} \hat{r}; \quad \frac{d\epsilon_{\text{kin}}}{dt} = \frac{\alpha}{r^2} u \cdot \hat{r}; \quad \alpha = ek. \quad (2.113) \]

The angular momentum is conserved:

\[ J = x \times p; \quad \frac{dJ}{dt} = u \times x + \frac{dp}{dt} = u \times \frac{m u}{\sqrt{1 - \frac{u^2}{c^2}}} + \frac{\alpha}{r} \hat{r} \times \hat{r} = 0. \quad (2.114) \]

Without loss of generality, we assume that

\[ J = J \hat{z}; \quad J > 0. \quad (2.115) \]

It is best to use cylindrical coordinates for this problem and we write

\[ x = r \hat{r} + z \hat{z}; \quad p = p_r \hat{r} + p_\phi \hat{\phi} + p_z \hat{z}, \quad (2.116) \]
where (accepting an abuse of notation) $r$ now stands for the length in the $x - y$ plane, $\hat{r}$ and $\hat{\phi}$ are the radial and angular direction in the $x - y$ plane and $(\hat{r}, \hat{\phi}, \hat{z})$ form a set of right-handed orthonormal vectors at every point in space. Then,

$$J = -zp_\phi \dot{r} + (zp_r - rp_z) \dot{\phi} + rp_\phi \dot{z} = J \hat{z}. \quad (2.117)$$

We therefore conclude

$$p_\phi > 0; \Rightarrow z = 0; \Rightarrow p_z = 0, \quad (2.118)$$

and Eq. $(2.116)$ reduces to

$$x = r \hat{r}; \quad p = p_r \hat{r} + p_\phi \hat{\phi}, \quad (2.119)$$

with the conclusion that the motion of the particle is restricted to the $x - y$ plane.

We need to work out time derivatives of $\dot{r}$ and $\dot{\phi}$. To this end, we note that

$$\dot{r} = \cos \phi \, \hat{i} + \sin \phi \, \hat{j}; \quad \dot{\phi} = -\sin \phi \, \hat{i} + \cos \phi \, \hat{j}. \quad (2.120)$$

Then

$$\frac{d\hat{r}}{dt} = \left( -\sin \phi \, \hat{i} + \cos \phi \, \hat{j} \right) \dot{\phi} = \hat{\phi}; \quad \frac{d\hat{\phi}}{dt} = -\left( \cos \phi \, \hat{i} + \sin \phi \, \hat{j} \right) \dot{\phi} = -\hat{\phi}; \quad \dot{\phi} = \frac{d\phi}{dt} \quad (2.121)$$

Therefore,

$$u = \frac{d(r \hat{r})}{dt} = \dot{r} \hat{r} + r \hat{\phi}; \quad u \cdot u = \dot{r}^2 + r^2 \dot{\phi}^2; \quad \dot{r} = \frac{dr}{dt} \quad (2.122)$$

The expression for the angular momentum becomes

$$p = \frac{m \left( \dot{r} \hat{r} + r \dot{\phi} \hat{\phi} \right)}{\sqrt{1 - r^2 + r^2 \dot{\phi}^2}}, \quad (2.123)$$

The expression for the angular momentum becomes

$$J = \frac{m \dot{r}^2 \hat{\phi}}{\sqrt{1 - r^2 + r^2 \dot{\phi}^2}}, \quad (2.124)$$

and therefore $J > 0$ implies that $\dot{\phi} > 0$.

Focussing on the equation for the kinetic energy in Eq. $(2.113)$, we can rewrite it using Eq. $(2.122)$ as

$$\frac{dE_{\text{kin}}}{dt} = \frac{\alpha}{\dot{r}^2} \dot{r} = -\frac{d}{dt} \frac{\alpha}{\dot{r}} \Rightarrow E_{\text{kin}} + \frac{\alpha}{\dot{r}} = E, \quad (2.125)$$

and the total energy, $E$, is a constant of motion. Note that $E$ has to be greater than zero if $\alpha > 0$ since the kinetic energy is always positive. Also note that if $r = \infty$ is a point in the motion of the particle, $E$ has to be greater than zero since the kinetic energy is always positive. In order to obtain the equation for orbits, $r(\phi)$, we work toward eliminating $\dot{r}$ and $\dot{\phi}$ in favor of

$$r' = \frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} \quad (2.126)$$

We first rewrite Eq. $(2.124)$ as an equation for $d\phi$ in terms of $r'$:

$$J^2 - \frac{J^2}{c^2} r^2 = \left( m^2 r^4 + \frac{J^2}{c^2} r^2 \right) \dot{\phi}^2 \Rightarrow \frac{J^2}{\dot{\phi}^2} = m^2 r^4 + \frac{J^2}{c^2} r^2 + \frac{J^2}{c^2} r'^2, \quad (2.127)$$

and we have

$$\dot{\phi} = \frac{Jc}{\sqrt{m^2 r^4 + J^2 r'^2 + J^2 r'^4}}, \quad (2.128)$$

where we have used the fact that $\dot{\phi} > 0$. From Eq. $(2.122)$, we have

$$u \cdot u = \dot{\phi}^2 \left( r^2 + r'^2 \right) = \frac{J^2 c^2 (r^2 + r'^2)}{m^2 r^4 + J^2 r'^2 + J^2 r'^4}; \quad 1 - \frac{u \cdot u}{c^2} = \frac{m^2 c^2 r^4}{m^2 r^4 + J^2 r'^2 + J^2 r'^4}. \quad (2.129)$$
Inserting this into the expression for the kinetic energy in Eq. (2.125), we obtain

\[ \frac{c}{r^2} \sqrt{m^2 c^2 r^4 + J^2 r^2 + J^2 r'^2 + \frac{\alpha}{r}} = E. \] (2.130)

and we obtain the equation for \( r' \) as

\[ r'^2 = \frac{r^2}{J^2 c^4} (E r - \alpha)^2 - \frac{m^2 c^4}{J^2} r^2 = \frac{E^2 - m^2 c^4}{J^2 c^2} r^4 - \frac{2E \alpha}{J^2 c^2} r^3 + \frac{\alpha^2 - J^2 c^2}{J^2 c^2} r^2. \] (2.131)

The right-hand side of this equation is quartic in \( r \) and we can reduce the order of the polynomial to quadratic by setting

\[ r = \frac{1}{\chi}; \quad r' = -\frac{\chi'}{\chi^2}. \] (2.132)

The equation for \( \chi' \) is

\[ \chi'^2 = \frac{E^2 - m^2 c^4}{J^2 c^2} - \frac{2E \alpha}{J^2 c^2} \chi + \frac{\alpha^2 - J^2 c^2}{J^2 c^2} \chi^2. \] (2.133)

The solution to the above equation proceeds differently depending upon the sign of \( \alpha \).

2.5.1 \( J = |\alpha| \)

In this case, Eq. (2.133) reduces to

\[ \chi'^2 = \frac{E^2 - m^2 c^4}{\alpha^2} - \frac{2E}{\alpha} \chi. \] (2.134)

If we set

\[ \psi = \frac{E^2 - m^2 c^4}{\alpha^2} - \frac{2E}{\alpha} \chi, \] (2.135)

then

\[ \psi' = -\frac{2E}{\alpha} \chi' \] (2.136)

and Eq. (2.134) reduces to

\[ \frac{\alpha^2}{4E^2} \psi'^2 = \psi. \] (2.137)

The solution is

\[ \psi(\phi) = \frac{E^2}{\alpha^2} \phi^2, \] (2.138)

where we have set \( \psi(0) = 0 \). This does not imply that \( \phi = 0 \) has to be in the path of the particle. Upon using Eq. (2.135) and Eq. (2.132), we arrive at

\[ \frac{1}{r(\phi)} = \frac{E^2 - m^2 c^4}{2E \alpha} - \frac{E}{2\alpha} \phi^2. \] (2.139)

Let us first consider the case of \( \alpha > 0 \) which implies that \( E > 0 \) from Eq. (2.125). Therefore the second term is positive for all values of \( \phi \). This implies that \( E \geq mc^2 \) since we require \( r(\phi) > 0 \) for all values of \( \phi \). Given a value for \( E \geq mc^2 \), we conclude that

\[ \phi^2 \leq \frac{E^2 - m^2 c^4}{E^2} \implies \phi \in \left[ -\sqrt{\frac{E^2 - m^2 c^4}{E}}, \sqrt{\frac{E^2 - m^2 c^4}{E}} \right]. \] (2.140)

The physical picture of the particle’s motion is one of scattering from a repulsive center. The particle is at \( r = \infty \) at the two ends of the allowed region for \( \phi \). The particle is at its closest approach when \( \phi = 0 \) and the value is given by

\[ r_{\text{min}} = \frac{2E \alpha}{E^2 - m^2 c^4}. \] (2.141)

Let us now consider the case of \( \alpha < 0 \). We rewrite Eq. (2.139) as

\[ \frac{1}{r(\phi)} = \frac{m^2 c^4 - E^2}{2E |\alpha|} + \frac{E}{2|\alpha|} \phi^2. \] (2.142)
For this case, we start with \( E > mc^2 \). Then

\[
\phi > \frac{\sqrt{E^2 - m^2c^4}}{E}. \tag{2.143}
\]

If \( \phi > \frac{\sqrt{E^2 - m^2c^2}}{E} \), the particle is at infinity and as \( \phi \) increases \( r \) decreases and reaches zero as \( \phi \) goes to infinity. That is to say, the particle spirals in from infinity and eventually reaches \( r = 0 \). If \( 0 < E < mc^2 \), then the particle has maximum value of \( r_{\text{max}} \) given by

\[
r_{\text{max}} = r(0) = \frac{2E \alpha}{m^2c^4 - E^2}. \tag{2.144}
\]

A particle that starts from there spirals in ans eventually reaches zero as \( \phi \to \infty \). The difference between \( E > mc^2 \) and \( 0 < E < mc^2 \) is evident when we try to reverse the motion and start the motion at \( r = 0 \). If \( E > mc^2 \), the particle escapes to \( r = \infty \) and if \( 0 < E < mc^2 \), the particle reaches \( r_{\text{max}} \) and gets pulled back by the attractive center. If \( E < 0 \), it is best to rewrite Eq. (2.139) as

\[
\frac{1}{r(\phi)} = \frac{E^2 - m^2c^4}{2|E\alpha|} - \frac{|E|}{2|\alpha|} \phi^2. \tag{2.145}
\]

We see that \(-mc^2 < E < 0\) is not allowed since the RHS is always negative. We also see that \( E < -mc^2 \) is not allowed since \( r = \infty \) cannot be in the path if \( E < 0 \).

### 2.5.2 \( Jc > |\alpha| \)

In this case, Eq. (2.133) can be rewritten as

\[
\chi^2 = \frac{J^2c^2 - \alpha^2}{J^2c^2} \left[ -\chi^2 - \frac{2E \alpha}{J^2c^2 - \alpha^2} \chi + \frac{E^2 - m^2c^4}{J^2c^2 - \alpha^2} \right]
= \frac{J^2c^2 - \alpha^2}{J^2c^2} \left[ -\chi^2 - \frac{E \alpha}{J^2c^2 - \alpha^2} \chi + \frac{E^2 - m^2c^4}{J^2c^2 - \alpha^2} \right]. \tag{2.146}
\]

Since the LHS is positive definite, we conclude that

\[
E^2 > m^2c^4 \frac{J^2c^2 - \alpha^2}{J^2c^2}, \tag{2.147}
\]

for a solution to exist and the solution is

\[
\frac{1}{r(\phi)} = -\frac{E \alpha}{J^2c^2 - \alpha^2} \pm c \sqrt{\frac{E^2J^2 - m^2c^4(J^2c^2 - \alpha^2)}{(J^2c^2 - \alpha^2)^2}} \cos \left[ \sqrt{\frac{J^2c^2 - \alpha^2}{J^2c^2}} \phi \right]. \tag{2.148}
\]

We can set the \( \pm \) sign to be positive without loss of generality since a negative sign can be converted to a positive sign by the change of variable

\[
\phi \to \pi \sqrt{\frac{J^2c^2}{J^2c^2 - \alpha^2}} + \phi. \tag{2.149}
\]

In order to analyze the various possible orbits, let us look the two possible cases:

1. \( E \alpha > 0 \): In order for the solution to exist, we require

\[
c^2 \left[ E^2J^2 - m^2c^4(J^2c^2 - \alpha^2) \right] > E^2 \alpha^2 \quad \Rightarrow \quad E^2 > m^2c^4. \tag{2.150}
\]

   The allowed values of \( \phi \) are

\[
\cos \left[ \sqrt{\frac{J^2c^2 - \alpha^2}{J^2c^2}} \phi \right] > \frac{E \alpha}{c \sqrt{E^2J^2 - m^2c^4(J^2c^2 - \alpha^2)}}. \tag{2.151}
\]

   Therefore, the physics is that of scattering with \( r = \infty \) in the path of the particle. Therefore, we require \( E > mc^2 \) and \( \alpha > 0 \) implying that this case only holds for the repulsive interaction.
2.5. ONE PHYSICAL EXAMPLE OF A CONSTANT NON-UNIFORM ELECTROMAGNETIC FIELD

2. \( E\alpha < 0 \): This can only hold for \( \alpha < 0 \) and \( E > 0 \) since \( \alpha > 0 \) requires \( E > 0 \). Here we have two possibilities:

(a) 

\[
c^2 [E^2 J^2 - m^2 c^2 (J^2 c^2 - \alpha^2)] > E^2 \alpha^2 \quad \Rightarrow \quad E > mc^2.
\]

The allowed values of \( \phi \) are

\[
\cos \left[ \sqrt{\frac{J^2 c^2 - \alpha^2}{J^2 c^2}} \phi \right] > \frac{|E\alpha|}{c \sqrt{E^2 J^2 - m^2 c^2 (J^2 c^2 - \alpha^2)}}.
\]

Therefore, the physics is that of scattering with \( r = \infty \) in the path of the particle.

(b) 

\[
c^2 [E^2 J^2 - m^2 c^2 (J^2 c^2 - \alpha^2)] < E^2 \alpha^2 \quad \Rightarrow \quad 0 < E < mc^2.
\]

Here the particle is in a bounded orbit due to an attractive interaction with

\[
\frac{1}{r(\phi)} \in \left[ \frac{|E\alpha|}{J^2 c^2 - \alpha^2} + c \sqrt{\frac{E^2 J^2 - m^2 c^2 (J^2 c^2 - \alpha^2)}{(J^2 c^2 - \alpha^2)}}, \frac{|E\alpha|}{J^2 c^2 - \alpha^2} - c \sqrt{\frac{E^2 J^2 - m^2 c^2 (J^2 c^2 - \alpha^2)}{(J^2 c^2 - \alpha^2)}} \right].
\]

- **5 points: Problem 2.18:** Is it a closed orbit? Provide a logical explanation for your answer. You may use plots if you wish, but note that plots do not provide a proof.

2.5.3 \( Jc < |\alpha| \)

- **20 points: Problem 2.19:** Derive the equation of the path, \( r(\phi) \), for this case and discuss all possibilities.
Chapter 3

Free electromagnetic fields

We have treated the electromagnetic field as external in the previous chapter. We have to write down an action for the free electromagnetic field just like we wrote down an action for the free particle in order to derive the equations of motion for the electromagnetic field. In the case of the free particle, we minimized the distance. In order to do the same for an electromagnetic field, we will need to introduce Einstein’s general theory of gravity and deduce the action for the free electromagnetic field. Instead, we will simply state the action by assuming it is a scalar and quadratic in the electromagnetic field.

3.1 Action for the electromagnetic field

The electromagnetic field exists everywhere in space and time and we will need the four-volume element

\[ d\Omega = dx^0 dx^1 dx^2 dx^3, \]

(3.1)
to define the action for the electromagnetic field. Since the determinant of a Lorentz transformation as defined in Eq. (1.40) is unity, it follows that the volume element is invariant under a Lorentz transformation and therefore a scalar.

Now we are ready to write down an action for the electromagnetic field itself, which we will denote by \( S_{em} \) and will be a function of the electromagnetic vector potential. We will require \( S_{em} \) to be gauge invariant since it can only depend on the physical electric and magnetic fields. As such, it will be a function of the electromagnetic field tensor. We will want the action to be a scalar and we have two candidates that depend quadratically on the electromagnetic field tensor as given by Eq. (2.55) and Eq. (2.56), namely,

\[ F_{ij}F^{ij} \quad \text{and} \quad \epsilon^{ijkl}F_{ij}F_{kl}. \]

(3.2)
The second one can be rewritten as

\[ \epsilon^{ijkl}\partial_i A_j \partial_k A_l = \partial_i \left[ \epsilon^{ijkl}A_j \partial_k A_l \right]. \]

(3.3)
Since it is a total derivative, integrating over the space and time results in a boundary term and therefore cannot affect the equations of motion. Therefore, the action for the electromagnetic field is

\[ S_{em} = -\frac{1}{16\pi c} \int d\Omega \, F_{ij}F^{ij}, \]

(3.4)
and the coefficient in front is fixed such that the resulting equations of motion take on the standard form.

3.2 Equations of motion for the free electromagnetic field

Consider the variation of \( S_{em} \) under \( A_j \rightarrow A_j + \delta A_j \) and we will use Eq. (2.43), namely,

\[ F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}. \]

(3.5)
We have

\[ \delta S_{em} = -\frac{1}{16\pi c} \int d\Omega \, \delta F_{ij}F^{ij} - \frac{1}{16\pi c} \int d\Omega \, F_{ij} \delta F^{ij} = -\frac{1}{8\pi c} \int d\Omega \, \delta F_{ij}F^{ij}. \]
\[ \frac{1}{8\pi c} \int d\Omega \left[ \frac{\partial \mathbf{A}_i}{\partial x^j} - \frac{\partial \mathbf{A}_j}{\partial x^i} \right] = \frac{1}{4\pi c} \int d\Omega \left[ \mathbf{A}_i \frac{\partial F^{ij}}{\partial x^j} - \mathbf{A}_j \frac{\partial F^{ij}}{\partial x^i} \right] \]

interchanged \( i \leftrightarrow j \) in the second term and used \( F^{ij} = -F^{ji} \) (3.6)

The equations of motion for the free electromagnetic field become

\[ \frac{\partial F^{ij}}{\partial x^i} = 0. \] (3.7)

The above equations, using Eq. (2.53), in terms of \( \mathbf{E} \) and \( \mathbf{H} \) fields are

\[ i = 0 \Rightarrow \nabla \cdot \mathbf{E} = 0; \quad i = 1, 2, 3 \Rightarrow \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}. \] (3.8)

These are the two equations with physics content. The other two equations

\[ \nabla \cdot \mathbf{H} = 0; \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} = 0 \] (3.9)

are a trivial mathematical consequence of the definitions of \( \mathbf{E} \) and \( \mathbf{H} \) in terms of vector potential as given in Eq. (2.33).

- 10 points: Problem 3.1: Show that

\[ T_{ikl} = \frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{il}}{\partial x^k} + \frac{\partial F_{lk}}{\partial x^i} \] (3.10)

is fully anti-symmetric and also show that

\[ T_{ikl} = 0 \] (3.11)

simply follows from the definition of the electromagnetic field tensor in Eq. (2.43). Show that the above equations are same as Eq. (3.9).

3.3 Noether’s theorem – One particular case

The scalar \( F_{ij} F^{ij} \) used in the action, \( S_{\text{em}} \) is referred to as the Lagrangian density. In this section, we will show in general that if an action defined according to

\[ S = \int dt dV \mathcal{L} \left( \phi, \frac{\partial \phi}{\partial x^i} \right) \] (3.12)

has no explicit dependence on the coordinates, then there exists quantities that are conserved (remain invariant) along the equations of motion for the fields \( \phi_k \). The index \( k \) on the field simply labels the different fields that contribute to the Lagrangian and does not necessarily refer to a Lorentz index.

Let us reiterate the derivation of the equations of motion for the fields, \( \phi_k \) by considering a variation of \( S \) under \( \phi_k \rightarrow \phi_k + \delta \phi_k \). We have

\[ \delta S = \int dt dV \left[ \frac{\partial \mathcal{L}}{\partial \phi_i} \delta \phi_i + \frac{\partial \mathcal{L}}{\partial \phi_k} \frac{\partial \phi_k}{\partial x^l} \delta \phi_l \right] \]

\[ = \int dt dV \left[ \frac{\partial \mathcal{L}}{\partial \phi_i} \delta \phi_i + \frac{\partial \left\{ \frac{\partial \mathcal{L}}{\partial \phi_i} \right\}}{\partial x^m} \delta \phi_l \right], \quad (3.13) \]

where we have used integration by parts and dropped boundary terms. This results in the Euler-Lagrange equations,

\[ \frac{\partial \mathcal{L}}{\partial \phi_i} = \frac{\partial \left\{ \frac{\partial \mathcal{L}}{\partial \phi_i} \right\}}{\partial x^m}. \] (3.14)
3.3. NOETHER’S THEOREM – ONE PARTICULAR CASE

We make the observation that the above derivation will not change if $\mathcal{L}$ had in addition and explicit dependence on the coordinates, $x_l$ since we only studied the extremization under a variation of the fields.

Let us now consider the variation of $\mathcal{L}$ under $x_p$ along the equations of motion for the fields:

\[
\frac{\partial \mathcal{L}}{\partial x^p} = \frac{\partial \mathcal{L}}{\partial \phi_l} \frac{\partial \phi_l}{\partial x^p} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi_l}{\partial x^m}} \frac{\partial \phi_l}{\partial x^m} \frac{\partial x^m}{\partial x^p} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi_l}{\partial x^m}} \frac{\partial \phi_l}{\partial x^p} \frac{\partial x^p}{\partial x^m} = \frac{\partial}{\partial x^m} \left[ \frac{\partial \mathcal{L}}{\partial \phi_l} \frac{\partial \phi_l}{\partial x^p} \right].
\]

(3.15)

Note that we have used the equations of motion as given by Eq. (3.14) in going from the first line to the second line. The above result can be recast as

\[
\frac{\partial T^m_p}{\partial x^m} = 0; \quad (3.16)
\]

where

\[
T^m_p = \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi_l}{\partial x^m}} \frac{\partial \phi_l}{\partial x^p} - \mathcal{L} \delta^m_p.
\]

(3.17)

Defining

\[
T^{mp} = g^{pq}T^m_q = g^{pq} \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi_l}{\partial x^m}} \frac{\partial \phi_l}{\partial x^q} - \mathcal{L} g^{mp},
\]

we can write the conservation equations as

\[
\frac{\partial T^{mp}}{\partial x^m} = 0. \quad (3.19)
\]

We note that we are free to change

\[
T^{mp} \rightarrow T^{mp} + \frac{\partial \psi^k_{mp}}{\partial x^k}; \quad \psi^k_{mp} = -\psi^{mkp}
\]

(3.20)

and still satisfy the conservation equations. We will address a choice of the tensor $\psi^k_{mp}$ for the specific case of free electromagnetic fields in the next section. For now, we will assume that we have used this freedom and made $T^{mp}$ into a symmetric tensor.

We proceed to provide a physical description of the conservation equations with an interpretation of the various components of the tensor, $T^{mp}$, assuming it is symmetric. For $p = 0$, Eq. (3.19) reduces to

\[
\frac{1}{c} \frac{\partial T^{00}}{\partial t} + \frac{\partial T^{10}}{\partial x} + \frac{\partial T^{20}}{\partial y} + \frac{\partial T^{30}}{\partial z} = 0.
\]

(3.21)

The above conservation equation defines the quantity,

\[
(\epsilon, c\mathbf{p}) = (T^{00}, T^{10}, T^{20}, T^{30})
\]

(3.22)

as the energy-momentum density and

\[
\frac{\partial \epsilon}{\partial t} + c^2 \nabla \cdot \mathbf{p} = 0.
\]

(3.23)

For $p = 1$, Eq. (3.19) reduces to (assuming the symmetric property)

\[
\frac{1}{c} \frac{\partial T^{10}}{\partial t} + \frac{\partial T^{11}}{\partial x} + \frac{\partial T^{21}}{\partial y} + \frac{\partial T^{31}}{\partial z} = 0 \Rightarrow \frac{\partial p_x}{\partial t} = -\frac{\partial T^{11}}{\partial x} - \frac{\partial T^{21}}{\partial y} - \frac{\partial T^{31}}{\partial z}.
\]

(3.24)

The left hand side is the rate of change of momentum density in the $x$ direction and therefore the right hand side is interpreted as the $x$-component of the internal force density and

\[
\sigma_{xx} = -T^{11}; \quad \sigma_{yx} = -T^{21}; \quad \sigma_{zx} = -T^{31}
\]

(3.25)
are components of stress tensor. Extending this to $p = 1$ and $p = 2$, we have the following physical interpretation for the energy-momentum-stress tensor, $T^{mp}$:

$$T^{mp} = \begin{pmatrix}
\epsilon & c \rho_x & c \rho_y & c \rho_z \\
c \rho_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\
c \rho_y & -\sigma_{xy} & -\sigma_{yy} & -\sigma_{yz} \\
c \rho_z & -\sigma_{xz} & -\sigma_{yz} & -\sigma_{zz}
\end{pmatrix}.$$  \hspace{1cm} (3.26)

### 3.4 Energy-momentum-stress tensor for the free electromagnetic field

Starting from the Lagrangian density as defined in Eq. (3.4),

$$\mathcal{L}(A_k, \frac{\partial A_k}{\partial x^l}) = -\frac{1}{16\pi} F_{lq} F^{lq},$$

and using the expression for $T^{mp}$ in Eq. (3.18), we arrive at

$$T^{mp} = -\frac{1}{4\pi} g^{pq} F_{ml} \frac{\partial A_l}{\partial x^p} + \frac{1}{16\pi} g^{mp} F_{lq} F^{lq}.$$  \hspace{1cm} (3.28)

In order to make this symmetric (the first term is not symmetric) using the freedom in Eq. (3.20), we note that

$$\frac{1}{4\pi} g^{pq} F_{ml} \frac{\partial A_l}{\partial x^p} = \frac{1}{4\pi} g^{mp} F_{lq} F^{lq}. \hspace{1cm} (3.29)$$

We have used the equations of motion for the free electromagnetic field as given by Eq. (3.7) in arriving at the last inequality. Noting that $\psi^{mp} = F_{ml} A^p$ satisfies the required property in Eq. (3.20), we conclude that we can add the above expression to $T^{mp}$ and arrive at the expression for the energy-momentum-stress tensor for the free electromagnetic field,

$$T^{mp} = \frac{1}{4\pi} \left[ g^{pq} F_{ml} + \frac{1}{4} g^{mp} F_{lq} F^{lq} \right] F_{lq}.$$  \hspace{1cm} (3.30)

**10 points:** Problem 3.2: Show that the energy-momentum-stress tensor is symmetric and traceless, namely,

$$T^{mp} = T^{pm}, \quad T^{mp} g_{pm} = 0.$$  \hspace{1cm} (3.31)

The energy density for the free electromagnetic field is

$$\epsilon = T^{00} = \frac{1}{4\pi} \left[ g^{0q} F_{0l} F^{0l} + \frac{1}{4} g^{00} F_{lq} F^{lq} \right] F_{lq} = \frac{1}{4\pi} F_{0l} F^{0l} + \frac{1}{4} F^{lq} F_{lq} = \frac{1}{8\pi} \left[ \mathbf{E} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{H} \right].$$  \hspace{1cm} (3.32)

We have used Eq. (2.44) and Eq. (2.53) in arriving at the last equality. The momentum density vector for the free electromagnetic field is

$$p_i = \frac{1}{c} T^{0i} = \frac{1}{4\pi c} g^{0q} F_{0l} F_{lq} = -\frac{1}{4\pi c} F_{0l} F_{li} \quad \Rightarrow \quad \mathbf{p} = \frac{1}{4\pi c} \mathbf{E} \times \mathbf{H}. \hspace{1cm} (3.33)$$

We have used Eq. (2.44) and Eq. (2.53) in arriving at the last equality. The conservation equation given by Eq. (3.23) becomes

$$\frac{\partial}{\partial t} \left( \frac{\mathbf{E} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{H}}{8\pi} \right) + \nabla \cdot \left( \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \right) = 0,$$  \hspace{1cm} (3.34)

and

$$\mathbf{S} = c^2 \mathbf{p} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}.$$  \hspace{1cm} (3.35)

is referred to as the Poynting vector.

**5 points:** Problem 3.3: Show that Eq. (3.34) directly follows from the equations of motion as given by Eq. (3.8) and Eq. (3.9).
3.5. WAVE EQUATIONS IN VACUUM

- **10 points: Problem 3.4:** Show that the components of the stress tensor are given by
  \[ \sigma_{\alpha\beta} = \frac{1}{4\pi} \left[ E_\alpha E_\beta + H_\alpha H_\beta - \frac{1}{2} \delta_{\alpha\beta} (E^2 + H^2) \right]; \quad \alpha, \beta = 1, 2, 3. \tag{3.36} \]

Show that
  \[ \frac{\partial p_\alpha}{\partial t} = \sum_{\beta=1}^{3} \frac{\partial \sigma_{\beta\alpha}}{\partial x^\beta}; \quad \alpha = 1, 2, 3 \tag{3.37} \]

directly follows from the equations of motion as given by Eq. (3.8) and Eq. (3.9).

3.5 Wave equations in vacuum

We can use the gauge degree of freedom discussed in Section 2.3 to simplify the expression for the equations of motion for the free electromagnetic field as given in Eq. (3.7). We use the Lorenz gauge,
  \[ \frac{\partial A^k}{\partial x^k} = 0; \quad \Rightarrow \quad \frac{1}{c} \frac{\partial \phi}{\partial t} + \nabla \cdot A = 0; \tag{3.38} \]

to fix the gauge degree of freedom. Under this gauge, Eq. (3.7) reduce to wave equations for all four components of the potential:
  \[ \frac{\partial^2}{\partial x^k \partial x^k} A^i = 0; \Rightarrow \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla \cdot \nabla \right) \begin{pmatrix} \phi \\
A_x \\
A_y \\
A_z \end{pmatrix} = 0. \tag{3.39} \]

One solution to this equation we will work with is
  \[ \phi = 0; \quad A_x = 0; \quad A_y(\xi); \quad A_z(\xi); \quad \xi = x - ct. \tag{3.40} \]
Since
  \[ \frac{d\xi}{dt} = \frac{dx}{dt} - c, \tag{3.41} \]

it follows that this solution corresponds to a wave propagating in the positive \( x \) direction obtained by setting \( \frac{d\xi}{dt} = 0 \). The associated electric and magnetic fields are
  \[ \mathbf{E} = (0, A'_y, A'_z); \quad \mathbf{H} = (0, -A'_z, A'_y), \tag{3.42} \]
where the prime denotes differentiation with respect to \( \xi \), the argument of the functions. Note that
  \[ \mathbf{E} \cdot \mathbf{H} = 0; \quad \mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} = \frac{c}{4\pi} \left( A'_y^2 + A'_z^2 \right) \mathbf{x}. \tag{3.43} \]

Of particular interest are periodic functions in the variable \( \xi \) with a period, \( T \):
  \[ A_y(\xi + cT) = A_y(\xi); \quad A_z(\xi + cT) = A_z(\xi); \tag{3.44} \]
such that the average over one period is zero;
  \[ \frac{1}{T} \int_{t_0}^{t_0+T} dt \ A_y(\xi) = \frac{1}{T} \int_{t_0}^{t_0+T} dt \ A_z(\xi) = 0, \tag{3.45} \]

and the average power over one period is
  \[ \frac{c}{4\pi T} \int_{t_0}^{t_0+T} dt \ \left( \{A'_y(\xi)\}^2 + \{A'_z(\xi)\}^2 \right) = s. \tag{3.46} \]

For future purposes, we will define the average
  \[ \frac{1}{T} \int_{t_0}^{t_0+T} dt \ \left( \{A_y(\xi)\}^2 + \{A_z(\xi)\}^2 \right) = \alpha^2. \tag{3.47} \]
3.6 Motion of a test particle in the field of a periodic electromagnetic wave

The relativistic equations of motion for a particle of mass $m$ and charge $e$ in the presence of an electromagnetic field are

$$\frac{dp}{dt} = eE + \frac{e}{c} u \times H, \quad p = \frac{mu}{\sqrt{1 - \frac{u \cdot u}{c^2}}}, \quad E = \frac{mc^2}{\sqrt{1 - \frac{u \cdot u}{c^2}}},$$  

(3.48)

where $p$ is the momentum vector and $E$ is the relativistic kinetic energy. Explicitly, using Eq. (3.42) and Eq. (3.41)

$$\frac{dp_x}{dt} = \frac{e}{c} (u_y A'_y + u_z A'_z); \quad \frac{dp_y}{dt} = e \left( 1 - \frac{u_x}{c} \right) A'_y = -\frac{e}{c} \frac{dA_y}{dt}; \quad \frac{dp_z}{dt} = e \left( 1 - \frac{u_x}{c} \right) A'_z = -\frac{e}{c} \frac{dA_z}{dt},$$  

(3.49)

The momentum of the particle in the directions perpendicular to the wave propagation are simple and given by

$$p_y + \frac{e}{c} A_y = \pi_y; \quad p_z + \frac{e}{c} A_z = \pi_z,$$  

(3.50)

where the conjugate momenta in the directions perpendicular to the wave propagation, $\pi_y$ and $\pi_z$ are constants of motion. If we impose the physically interesting condition that the average momentum over a period is zero;

$$\frac{1}{T} \int_{t_0}^{t_0 + T} p \, dt = 0;$$  

(3.51)

then Eq. (3.45) along with Eq. (3.50) implies $\pi_y = \pi_z = 0$ and

$$p_y = -\frac{e}{c} A_y; \quad p_z = -\frac{e}{c} A_z.$$  

(3.52)

In order to solve the momentum of the particle in the direction of the wave propagation, we start by writing the equation for $p_x$ in Eq. (3.49) using the first of the two equalities in Eq. (3.49) for $p_y$ and $p_z$ as

$$\frac{dp_x}{dt} = \frac{u_y dp_y}{c - u_x} + \frac{u_z dp_z}{c - u_x}.$$  

(3.53)

We can rewrite this equation using the expressions for momentum and kinetic energy in Eq. (3.48) as

$$\frac{E}{c} \frac{dp_x}{dt} = \frac{p_x \frac{dp_x}{dt}}{c} + \frac{p_y \frac{dp_y}{dt}}{c} + \frac{p_z \frac{dp_z}{dt}}{c} = \frac{1}{2} \frac{d}{dt} \left( \frac{p_x^2 + p_y^2 + p_z^2}{c^2} \right) = \frac{1}{2} \frac{dE}{c^2 dt},$$  

$$\frac{E}{c} \frac{dp_x}{dt} = \frac{1}{2} \frac{d}{dt} \left( \frac{p_x^2 + p_y^2 + p_z^2}{c^2} \right) = \frac{1}{2} \frac{dE}{c^2 dt},$$  

(3.54)

and we conclude that

$$\frac{E}{c} = p_x = \gamma$$  

(3.55)

is a constant of motion. If we now rewrite Eq. (3.53) using the expression for momentum in Eq. (3.48) as

$$\frac{dp_x}{dt} = \frac{p_x \frac{dp_x}{dt}}{c} + \frac{p_z \frac{dp_z}{dt}}{c - p_x},$$  

(3.56)

then we can write the solution for $p_x$ using Eq. (3.55) as

$$p_x = \frac{1}{2\gamma} \left( p_y^2 + p_z^2 \right) + k_x$$  

(3.57)
3.7 Monochromatic waves

and \( k_x \) is a constant to be determined. Using the solutions for \( p_y \) and \( p_z \) in Eq. (3.52) and the definition in Eq. (3.47), the condition for \( p_x \) in Eq. (3.51) fixes \( k_x \) as

\[
k_x = -\frac{e^2 \alpha^2}{2\gamma c^2}.
\]

We can rewrite the solution for \( p_x \) in Eq. (3.57) as

\[
p_x = \frac{e^2}{2c^2\gamma} (A_y^2 + A_z^2 - \alpha^2)
\]

We still need to determine the constant of motion, \( \gamma \). For this purpose, we use Eq. (3.55) in Eq. (3.57) and write

\[
\frac{\mathcal{E}}{c} - \gamma = \frac{1}{2\gamma} \left[ \frac{\mathcal{E}^2}{c^2} - m^2 c^2 - \left( \frac{\mathcal{E}}{c} - \gamma \right)^2 \right] - \frac{e^2 \alpha^2}{2\gamma c^2}
\]

Therefore,

\[
\gamma^2 = m^2 c^2 + \frac{e^2 \alpha^2}{c^2}.
\]

\[\text{5 points: Problem 3.5:} \quad \text{Prove the relation} \]

\[
p = -\gamma \frac{dr}{d\xi}.
\]

\[\text{10 points: Problem 3.6:} \quad \text{Derive the following solutions of} \ r(\xi):
\]

\[
x(\xi) = \frac{e^2}{2c^2\gamma} \left[ \alpha^2 \xi - \int_0^\xi (\{A_y(\xi')\}^2 + \{A_z(\xi')\}^2) d\xi' \right]
\]

\[
y(\xi) = \frac{e}{\gamma c} \int_0^\xi A_y(\xi') d\xi'
\]

\[
z(\xi) = \frac{e}{\gamma c} \int_0^\xi A_z(\xi') d\xi'
\]

3.7 Monochromatic waves

A monochromatic electromagnetic wave is one where the periodic function takes the form of \( \cos(k\xi) \) or \( \sin(k\xi) \). The wavelength \( \lambda \) is related to the wavenumber \( k \) by

\[
k = \frac{2\pi}{\lambda},
\]

and the frequency is

\[
\omega = kc.
\]

The most general expression for \( A_y(\xi) \) and \( A_z(\xi) \) become

\[
A_y(\xi) = A_{1y} \cos(k\xi) + A_{2y} \sin(k\xi); \quad A_z(\xi) = A_{1z} \cos(k\xi) + A_{2z} \sin(k\xi).
\]

\[\text{5 points: Problem 3.7:} \quad \text{Consider a rotation of the} \ y, z \text{plane in the counter-clockwise direction by} \ \theta. \text{Let the new coordinates be denoted by primes and let}
\]

\[
A'_y(\xi) = A_{1y}' \cos(k\xi) + A_{2y}' \sin(k\xi); \quad A'_z(\xi) = A_{1z}' \cos(k\xi) + A_{2z}' \sin(k\xi),
\]

be the components of the vector potential in the \( y' \) and \( z' \) directions. Show that

\[
A_{1y}' = A_{1y} \cos \theta + A_{1z} \sin \theta; \quad A_{2y}' = A_{2y} \cos \theta + A_{2z} \sin \theta
\]

\[
A_{1z}' = A_{1z} \cos \theta - A_{1y} \sin \theta; \quad A_{2z}' = A_{2z} \cos \theta - A_{2y} \sin \theta.
\]
CHAPTER 3. FREE ELECTROMAGNETIC FIELDS

- **10 points: Problem 3.8:** Show that one can always write
\[
A'_1 = A_{0y} \sin \alpha; \quad A'_2 = A_{0y} \cos \alpha; \quad A'_1 = A_{0z} \cos \alpha; \quad A'_2 = -A_{0z} \sin \alpha.
\] (3.69)
where
\[
\tan 2\theta = \frac{2(A_{1y}A_{1z} + A_{2y}A_{2z})}{A_{1y}^2 + A_{2y}^2 - A_{1z}^2 - A_{2z}^2}.
\] (3.70)

The conclusion from the above two problems is that we can write the potential for the most general monochromatic electromagnetic wave as
\[
A(\xi) = A_{0y} \sin(kx - \omega t + \alpha)y + A_{0z} \cos(kx - \omega t + \alpha)z.
\] (3.71)
Since
\[
\left( \frac{A_1(\xi)}{A_{0y}} \right)^2 + \left( \frac{A_z(\xi)}{A_{0z}} \right)^2 = 1,
\] (3.72)
the most general monochromatic electromagnetic wave is elliptically polarized. A limiting case is linear polarization when either \(A_{0y}\) or \(A_{0z}\) is zero.

- **5 points: Problem 3.9:** Show that the associated electric and magnetic field are
\[
E(\xi) = kA_{0y} \cos(kx - \omega t + \alpha)y - kA_{0z} \sin(kx - \omega t + \alpha)z
\]
\[
H(\xi) = kA_{0z} \sin(kx - \omega t + \alpha)y + kA_{0y} \cos(kx - \omega t + \alpha)z.
\] (3.73)
As expected, show that the electric field and magnetic field are perpendicular to each other at all points in space and time. Furthermore, show that the flux is
\[
S = \frac{k^2 c}{8\pi} [A_{0y}^2 + A_{0z}^2 + (A_{0y}^2 - A_{0z}^2) \cos(2(kx - \omega t + \alpha))] \mathbf{x}.
\] (3.74)
The energy flux is a constant everywhere in space and time if the wave is circularly polarized, namely, \(A_{0y} = A_{0z}\).

- **20 points: Problem 3.10:** Obtain the equations of motion of a test particle in the field of a monochromatic electromagnetic wave. Describe your results for the special cases of linearly and circularly polarized wave using plots of \(x(t), y(t)\) and \(z(t)\).

### 3.8 Electromagnetic waves in a homogeneous and isotropic medium

As we will see later in the course, Maxwell’s equations in a homogeneous and isotropic medium can be written as
\[
\nabla \cdot \mathbf{E} = 0; \quad \nabla \cdot \mathbf{H} = 0; \quad \nabla \times \mathbf{E} = -\frac{\mu}{c} \frac{\partial \mathbf{H}}{\partial t}; \quad \nabla \times \mathbf{H} = \frac{\epsilon}{c} \frac{\partial \mathbf{E}}{\partial t},
\] (3.75)
where \(\epsilon > 1\) and \(\mu > 1\) are the electric permittivity and the magnetic permeability of the medium.

In order to derive the wave equation directly for the electric field, we note that
\[
\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}; \quad v = \frac{c}{\sqrt{\mu \epsilon}} < c,
\] (3.76)
directly follows from Eq. (3.75). Using, \(\nabla \cdot \mathbf{E} = 0\), this reduces to
\[
\frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - (\nabla \cdot \nabla) \mathbf{E} = 0.
\] (3.77)
Identical equations can be derived for \(\mathbf{H}\). The monochromatic solutions with an arbitrary direction of propagation are
\[
\mathbf{E} = \mathbf{E}_0 e^{i(kr - \omega t)}; \quad \mathbf{H} = \mathbf{H}_0 e^{i(kr - \omega t)}; \quad k = kn; \quad \omega = kv,
\] (3.78)
and \(\mathbf{n}\) is the unit vector in the direction of propagation. Inserting these into Eq. (3.75) results in
\[
\mathbf{n} \cdot \mathbf{E}_0 = 0; \quad \mathbf{n} \cdot \mathbf{H}_0 = 0; \quad \sqrt{\epsilon} \mathbf{H}_0 = \sqrt{\mu} \mathbf{n} \times \mathbf{E}_0.
\] (3.79)

In order to study the problem of reflection and refraction, we will need to consider two media with a boundary separating them. We will only consider the simplest case where the boundary is specified by the infinite plane, \(z = z_0\). As we will see later, the boundary conditions are given by

\footnote{It is sufficient to write the wave form as a complex phase for the discussion that follows.}
1. Tangential components: \( E_x, E_y, H_x \) and \( H_y \) are continuous.

2. Normal components: \( \epsilon E_z \) and \( \mu H_z \) are continuous.

Let us assume that we have a single boundary at \( z = 0 \) separating two media with \( \epsilon_1, \mu_1 \) and \( \epsilon_2, \mu_2 \) as their permittivities and permeabilities. Let us assume that we have a monochromatic incident wave in the first medium that results in a reflected wave in the first medium and a transmitted wave in the second medium. Since the origin is at the boundary and since the sum of the reflected wave and the transmitted wave must be equal to the incident wave at the origin at all times, it follows that \( \omega \) must be the same for all three waves. We therefore write

\[
E_i e^{i(k_i r - \omega t)}; \quad H_i e^{i(k_i r - \omega t)}
\]

for the incident wave,

\[
E_t e^{i(k_t r - \omega t)}; \quad H_t e^{i(k_t r - \omega t)}
\]

for the transmitted wave and

\[
E_r e^{i(k_r r - \omega t)}; \quad H_r e^{i(k_r r - \omega t)}
\]

for the reflected wave. Furthermore, we will assume that our coordinates have been chosen such that

\[
k_t = k_1 n_t; \quad n_t = \sin \theta_t x + \cos \theta_t z; \quad k_1 = \frac{\omega}{c} \sqrt{\mu_1 \epsilon_1}.
\]

Since, the sum of the reflected wave and the transmitted wave must be equal to the incident wave everywhere on the boundary, \( z = 0 \), it follows that

\[
k_r = k_2 n_r; \quad n_r = \sin \theta_r x + \cos \theta_r z; \quad k_2 = \frac{\omega}{c} \sqrt{\mu_2 \epsilon_2},
\]

for the transmitted wave and

\[
k_r = k_1 n_r; \quad n_r = \sin \theta_r x - \cos \theta_r z,
\]

for the reflected wave, with

\[
k_1 \sin \theta_t = k_2 \sin \theta_t = k_1 \sin \theta_r.
\]

These are the two standard laws of reflection and refraction of light:

1. **Law of reflection:** Angle of incidence is equal to the angle of reflection;

\[
\theta_r = \theta_t.
\]

2. **Snell’s law of refraction:**

\[
\sqrt{\frac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2}} \sin \theta_t = \sqrt{\frac{\mu_2 \epsilon_2}{\mu_1 \epsilon_1}} \sin \theta_t \Leftrightarrow \frac{\sin \theta_t}{v_1} = \frac{\sin \theta_t}{v_2}.
\]

Assuming \( v_1 < v_2 \), we see that a transmitted wave exists only if the incident angle satisfies

\[
\theta_t < \theta_c; \quad \theta_c = \sin^{-1} \frac{v_1}{v_2}.
\]

If \( \theta_t > \theta_c \), we have the phenomenon of **total internal reflection**.

We define the unit vectors,

\[
p_i = n_i \times y = -\cos \theta_t x + \sin \theta_t z; \quad p_t = n_t \times y = -\cos \theta_t x + \sin \theta_t z; \quad p_r = n_r \times y = \cos \theta_r x + \sin \theta_r z.
\]

and write

\[
E_i = E_{iy} y + E_{ip} p_i; \quad E_t = E_{iy} y + E_{ip} p_t; \quad E_r = E_{ry} y + E_{rp} p_r.
\]

The amplitudes of the magnetic field that follow from Eq. \((3.79)\) are

\[
H_i = \sqrt{\frac{\mu_1}{\epsilon_1}} [E_{ip} p_i - E_{ip} y]; \quad H_t = \sqrt{\frac{\mu_2}{\epsilon_2}} [E_{ip} p_t - E_{ip} y];
\]
\[ H_y = \sqrt{\frac{\epsilon_1}{\mu_1}} (E_{ry} - E_{rp}) \cos \theta_i; \]  

(3.92)

Continuity of tangential components using the law of reflection result in 

**x** direction:

\[ E : \ E_{ip} - E_{rp} \cos \theta_i = E_{tp} \cos \theta_t; \]  

(3.93)

\[ H : \ \sqrt{\frac{\epsilon_1}{\mu_1}} (E_{iy} - E_{ry}) \cos \theta_i = \sqrt{\frac{\epsilon_2}{\mu_2}} E_{ty} \cos \theta_t. \]  

(3.94)

**y** direction:

\[ E : \ E_{iy} + E_{ry} = E_{ty}; \]  

(3.95)

\[ H : \ \sqrt{\frac{\epsilon_1}{\mu_1}} (E_{ip} + E_{rp}) = \sqrt{\frac{\epsilon_2}{\mu_2}} E_{tp}. \]  

(3.96)

Continuity of normal component in the **z** direction using the law of reflection result in

\[ E : \ \epsilon_1 (E_{ip} + E_{rp}) \sin \theta_i = \epsilon_2 E_{tp} \sin \theta_t; \]  

(3.97)

\[ H : \ \sqrt{\mu_1 \epsilon_1} (E_{iy} + E_{ry}) \sin \theta_i = \sqrt{\mu_2 \epsilon_2} E_{ty} \sin \theta_t. \]  

(3.98)

Using Snell’s law of refraction, we can see that Eq. (3.97) is identical to Eq. (3.96) and Eq. (3.98) is identical to Eq. (3.95). Solving Eq. (3.93) and Eq. (3.96), we obtain

\[ \frac{E_{rp}}{E_{ip}} = \frac{2 \sqrt{\frac{\mu_1}{\mu_2}} \cos \theta_i}{\sqrt{\frac{\mu_1}{\mu_2} \cos \theta_i + \sqrt{\frac{\mu_1}{\mu_2} \cos \theta_t}}}, \quad \frac{E_{tp}}{E_{ip}} = \frac{2 \sqrt{\frac{\mu_2}{\mu_1}} \cos \theta_t}{\sqrt{\frac{\mu_2}{\mu_1} \cos \theta_i + \sqrt{\frac{\mu_2}{\mu_1} \cos \theta_t}}}. \]  

(3.99)

Solving Eq. (3.94) and Eq. (3.95), we obtain

\[ \frac{E_{ry}}{E_{iy}} = \frac{2 \sqrt{\frac{\mu_1}{\mu_2}} \cos \theta_i}{\sqrt{\frac{\mu_1}{\mu_2} \cos \theta_i + \sqrt{\frac{\mu_1}{\mu_2} \cos \theta_t}}}, \quad \frac{E_{ty}}{E_{iy}} = \frac{2 \sqrt{\frac{\mu_2}{\mu_1}} \cos \theta_t}{\sqrt{\frac{\mu_2}{\mu_1} \cos \theta_i + \sqrt{\frac{\mu_2}{\mu_1} \cos \theta_t}}}. \]  

(3.100)

**15 points: Problem 3.11:** What are the possible ways (conditions on \( \mu_1, \epsilon_1, \mu_2 \) and \( \epsilon_2 \) for there to be no reflected wave? Derive all necessary expressions and exhaustively discuss all possibilities. Explain how you might use this to convert elliptically polarized light to linearly polarized light.

**15 points: Problem 3.12:** Solve the problem when \( v_1 < v_2 \) and \( v_2 \sin \theta_i > v_1 \). Argue that one needs to write the transmitted electric and magnetic field as

\[ E_t e^{ik_z x - k_z z - i \omega t}, \quad H_t e^{ik_z x - k_z z - i \omega t}, \quad k_z > 0. \]  

(3.101)

Show that

\[ k_{tx} = k_i \sin \theta_i; \quad k_{tz} = \sqrt{k_i^2 \sin^2 \theta_i - k_z^2}. \]  

(3.102)

Work through the rest to arrive at the explicit expressions for the amplitudes of the reflected and transmitted waves for both the electric and magnetic field. Is there a transmitted wave? If so, describe its properties.
Chapter 4

The physics of interacting charges

Consider the physics of \( n \) point like charged particles with \( n \) being very large such that we can treat them using continuum variables. Based on what we have discussed in the previous chapters, we know

- the free action for the point particles (see Eq. (2.4));
- the free action for the electromagnetic field (see Eq. (3.4)); and
- and we also know the interaction term between each point particle and an electromagnetic field (see Eq. (2.57)).

We combined Eq. (2.4) and Eq. (2.57) to obtain the equations of motion for the particles. If we combine Eq. (3.4) and Eq. (2.57) we can derive the equations of motion for the electromagnetic field in the presence of particles. Since our primary interest is to study the motion of particles, it seems natural to eliminate the electromagnetic field from the equations of motion. But, it is in general useful not to take this viewpoint since a better physical understanding is usually gained by keeping the intermediate electromagnetic fields. It is best to move away from the notation of a discrete set of point particles to a continuum of charges before deriving the equations of motion for the electromagnetic field.

4.1 A continuum of charges

In preparation for studying the interaction of many charges, let us now consider many charges such that we can treat them using a continuum charge density, \( \rho(x, t) \). If we had a finite number of charges, \( e_i \) located at \( x_i(t) \) for \( i = 1, \ldots, n \), then

\[
\rho(x, t) = \sum_{i=1}^{n} e_i \delta(x - x_i(t)).
\]

Since the total charge will not change as a function of time (conservation of total charge), we have

\[
\int dx_1 dx_2 dx_3 \rho(x, t) = Q
\]

is a conserved quantity. The quantity \( dx_1 dx_2 dx_3 \) changes under a Lorentz transformation of the form in Eq. (1.40) and therefore we conclude that \( \rho(x, t) \) also much change under a Lorentz transformation and therefore cannot be a scalar.

From the invariance of \( d\Omega \) and the invariance of \( Q \), it follows that \( \rho \) must transform like the time component of a four vector. To make this explicit, we rewrite Eq. (4.2) using Eq. (3.1) as

\[
\int dx^0 dx^1 dx^2 dx^3 \rho(x, t) \frac{dx^1}{c dt} = Q dx^i \Rightarrow \int d\Omega \left[ \rho(x, t) \frac{dx^1}{c dt} \right] = Q dx^i.
\]

This implies that

\[
\left( \rho(x, t), \rho(x, t) \frac{u}{c} \right) = \frac{1}{c} j^i(x, t); \quad u = \frac{dx}{dt},
\]

transforms as a four vector and we refer to

\[
j^i(x, t) = \left( c\rho(x, t), j(x, t) \right) = \left( c\rho(x, t), \rho(x, t)u \right)
\]
as the four-vector current.

The interaction term in the Lagrangian for a single charge in Eq. (2.57) is

\[ S_{\text{int}} = -\int \frac{e}{c} A_i(x(t), t) dx^i \]  

(4.6)

where the integration is along the path of the particle, \(x(t)\). When we have more than one particles, we will have a term like the above for each particle where the integration is along the path of each particle. We can make this explicit by rewriting the above expression for the \(j\)th particle and summing over all \(j\) as follows:

\[ S_{\text{int}} = -\frac{1}{c} \int dV \rho A_i dx^i \]

(4.7)

where we have used Eq. (4.5) in the last line. The interaction term has a simple continuum interpretation: The electromagnetic vector potential at a particular point in space and time interacts with the four current associated with the continuum of moving charges at that point in space and time. This scalar interaction is integrated over the scalar four volume (all space and time).

### 4.2 Maxwell’s equations

In order to obtain the equations of motion for the electromagnetic field in the presence of a current four-vector, we need to minimize \(S_{\text{em}} + S_{\text{int}}\) viewed as a functional of \(A_i\) for a fixed \(j_i\). Using the result in Eq. (3.6) and using Eq. (4.7) we arrive that

\[ \delta S_{\text{em}} + \delta S_{\text{int}} = \frac{1}{c} \int d\Omega \left\{ \delta A_k \left[ \frac{1}{4\pi} \frac{\partial F^{jk}}{\partial x^j} - \frac{1}{c} j^k \right] \right\} \]  

(4.8)

The equations of motion for the electromagnetic field in the presence of a current four-vector is

\[ \frac{\partial F^{jk}}{\partial x^j} = \frac{4\pi}{c} j^k \]  

(4.9)

Therefore, Eq. (3.8) become

\[ k = 0 \Rightarrow \nabla \cdot E = 4\pi \rho; \quad k = 1, 2, 3 \Rightarrow \nabla \times H = \frac{1}{c} \frac{\partial E}{\partial t} + \frac{4\pi}{c} j \]  

(4.10)

and the other equations given by Eq. (3.9) remain the same.

### 4.3 Electromagnetic potentials for a general charge and current distribution

Given the charge and current distribution, we can solve for the electromagnetic potential, \(A(x, t)\) and \(\phi(x, t)\) using Eq. (4.9) in the Lorentz gauge given by Eq. (3.38). The equations are

\[ \partial_j \partial^j A^k = \frac{4\pi}{c} j^k \]  

(11.11)

Due to the linearity of these equations, it is sufficient to solve

\[ \partial_j \partial^j G = 4\pi \prod_{k=1}^{4} \delta(x_k - x'_k) = \frac{4\pi}{c} \delta(x - x')\delta(t - t') \]  

(4.12)
Therefore, we will concern ourselves only with the particular solution to this problem since we will assume that there is no background electromagnetic wave. This problem can be solved by writing the delta function in terms of its Fourier components:

\[
\delta(r_j) = \frac{1}{2\pi} \int dk_j \, e^{ik_j r_j}; \quad r_j = x_j - x_j'.
\] (4.13)

**Points: Problem 4.1:** This problem will lead you through a proof of Eq. (4.13). Consider a periodic one-dimensional region, \( x \in [-\frac{L}{2}, \frac{L}{2}] \). Let \( f(x) \) be an arbitrary periodic function in this region. We can expand \( f(x) \) in a Fourier series. That is to say, we consider specific functions,

\[
g_n(x) = e^{-\frac{2\pi\pi i nx}{L}}.
\] (4.14)

Show that \( n \) has to be an integer for \( g_n(x) \) to be a periodic function. Show that

\[
\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \, g_n(x)g_{n'}(x) = \begin{cases} \frac{L}{2} & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases}
\] (4.15)

Define

\[
\tilde{f}_n = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \, f(x)g_n(x).
\] (4.16)

Then show that

\[
f(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \tilde{f}_n g_n(x),
\] (4.17)

by inserting Eq. (4.17) in Eq. (4.16) and using Eq. (4.15). If \( f(x) = \delta(x) \), show that \( \tilde{f}_n = 1 \) for all \( n \). Finally show that

\[
\delta(x) = \lim_{L \to \infty} \frac{1}{L} \sum_{n=-\infty}^{\infty} \tilde{f}_n g_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx}.
\] (4.18)

The solution to Eq. (4.12) is

\[
G(r_j) = -\frac{1}{4\pi^3} \int d^3k \, \frac{e^{ik_j r_j}}{k_0 k^3} = -\frac{1}{4\pi^3} \int d^3k \, e^{-ikr} \int_{-\infty}^{\infty} dk_0 \, \frac{e^{ik_0 r_0}}{k_0^2 - k^2}.
\] (4.19)

In order perform the integral, we need to move the singularity away from the line of integration. We write,

\[
f_{\pm}(r_0, k) = \frac{1}{2k} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dk_0 e^{ik_0 r_0} \left( \frac{1}{k_0 - k \pm i\epsilon} - \frac{1}{k_0 + k \pm i\epsilon} \right); \quad \epsilon > 0,
\] (4.20)

and

\[
G_{\pm}(r_j) = -\frac{1}{4\pi^3} \int d^3k e^{-ikr} f_{\pm}(r_0, k).
\] (4.21)

In order to perform the three dimensional integral over \( k \), we use spherical polar coordinates and use the freedom in our choice of axes to set \( r \) in the \( z \) direction. Then \( k \cdot r = kr \cos \theta \) and \( d^3k = 2\pi k^2 \, dk \, d\cos \theta \). Performing the integral over \( \cos \theta \) we arrive at

\[
G_{\pm}(r_j) = \frac{i}{2\pi} \int_{0}^{\infty} dk' \left( e^{ik'r} - e^{-ik'r} \right) f_{\pm}(r_0, k).
\] (4.22)

We can evaluate \( f_{\pm}(r_0, k) \) by contour integration. If \( r_0 > 0 \) \( (r_0 < 0) \), we can evaluate \( f_{\pm}(r_0, k) \) by closing the contour in the upper-half (lower-half) plane. While evaluating \( f_{\pm}(r_0, k) \) \( f_{\pm}(r_0, k) \) we get a non-zero result if \( r_0 < 0 \) \( (r_0 > 0) \) and they are

\[
f_{\pm}(r_0, k) = \pm \frac{\pi i}{k} \theta(\mp r_0) \left( e^{ir_0 k} - e^{-ir_0 k} \right).
\] (4.23)

Therefore,

\[
G_{\pm}(r_j) = \pm \frac{1}{2\pi r} \int_{0}^{\infty} dk \left( e^{ik(r+r_0)} + e^{-ik(r+r_0)} - e^{ik(r-r_0)} - e^{-ik(r-r_0)} \right).
\]
\[ g = \pm \frac{1}{2\pi r} \int_{-\infty}^{\infty} dk \left( e^{ik(r+r_0)} - e^{ik(r-r_0)} \right) \]
\[ = \pm \frac{1}{r} \theta(\mp r_0) \left( \delta(r+r_0) - \delta(r-r_0) \right) \]
\[ = \frac{\delta(r \pm r_0)}{r}. \quad (4.24) \]

Explicitly,
\[ G_+(r, t' - t) = \frac{\delta(t' - (t + \frac{r}{c}))}{cr}; \quad G_-(r, t' - t) = \frac{\delta(t' - (t - \frac{r}{c}))}{cr}. \quad (4.25) \]

\[ G_- (G_+) \] is referred to as the retarded (advanced) Green’s function since the electromagnetic potential at time \( t \) is due to charges and currents at time \( t' = t - \frac{r}{c} \) \( (t' = t + \frac{r}{c}) \). For causal reasons, we will use the retarded Green’s function. Using the identity,
\[ s(x, t) = \int d^3x' dt' s(x', t') \delta(x - x') \delta(t - t'), \quad (4.26) \]
we arrive at the following expressions for the electromagnetic potential as the solution to Eq. (4.11) with \( G_- (r, t' - t) \) as the solution to Eq. (4.12):
\[ \phi(x, t) = \int d^3x' \rho(x', t - \frac{|x - x'|}{c}) \]
\[ A(x, t) = \int d^3x' j(x', t - \frac{|x - x'|}{c}). \quad (4.27) \]

The electromagnetic potential at a given time, \( t \), is due to the charge and current distribution at earlier times, \( t' < t \). The potential at a given point in space, \( x \), is due to charge and current distributions at all other points in space, \( x' \). For each choice of \( x' \), there is a corresponding \( t' \) given by
\[ t' = t - \frac{|x - x'|}{c}. \quad (4.28) \]

**• points: Problem 4.2:** The Feynman way: Define
\[ f_\pm (r_0, k) = \frac{1}{2k} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dk_0 e^{ik_0 r_0} \left( \frac{1}{k_0 - k \pm i\epsilon} - \frac{1}{k_0 + k \pm i\epsilon} \right); \quad \epsilon > 0, \quad (4.29) \]
instead of Eq. (4.20). Derive explicit expressions for \( G_\pm (x, t) \) starting from these two cases. How do your expressions differ from Eq. (4.25)? Compare your answers from a physics viewpoint in some detail. You may use the following identity without proof:
\[ \int_0^{\infty} e^{-ikx} dx = \frac{1}{ik}. \quad (4.30) \]

### 4.4 Lienard-Wiechert potentials

Let us focus on the case where we have a single charge \( e \) moving along a path given by \( r_e(t) \). Then, as per Eq. (4.1) and Eq. (4.5),
\[ \rho(x, t) = e\delta(x - r_e(t)); \quad j(x, t) = e v_e(t) \delta(x - r_e(t)); \quad v_e(t) = \frac{dr_e(t)}{dt}. \quad (4.31) \]
Using the solution, \( G_- (r, t' - t) \), and the identity in Eq. (4.26) we arrive at
\[ \phi(x, t) = e \int d^3x' dt' \delta(x' - r_e(t')) \frac{\delta(t' - t + \frac{|x - x'|}{c})}{|x - x'|}. \quad (4.32) \]
Upon performing the integral over all three spatial variables, we obtain
\[ \phi(x, t) = e \int dt' \frac{\delta(t' - t + \frac{|x - r_e(t')|}{c})}{|x - r_e(t')|}. \quad (4.33) \]
4.4. LIENARD-WIECHERT POTENTIALS

Let $\tau$ be the solution to the equation

$$\tau - t + \frac{|x - r_e(\tau)|}{c} = 0. \quad (4.34)$$

for a given $(x, t)$. In order to perform the integral over $t'$, we write

$$F(t') = t' - t + \frac{\mathbf{x} - r_e(t')}{c} = (t' - \tau)F(\tau) + \cdots; \quad F'(t') = \frac{dF(t')}{dt'} \bigg|_{(x, t)}. \quad (4.35)$$

An explicit differentiation gives us

$$F'(t') = 1 - \frac{(x - r_e(t')) \cdot \mathbf{v}_e(t')}{c|x - r_e(t')|}. \quad (4.36)$$

Performing the integral over $t'$ in Eq. (4.33), the scalar potential everywhere due to a charged particle, $e$, moving along the path $r_e(t)$ is

$$\phi(x, t) = \frac{ee}{c|x - r_e(\tau)| - (x - r_e(\tau)) \cdot \mathbf{v}_e(\tau)}, \quad (4.37)$$

where $\tau$ is the solution to Eq. (4.34).

- **points:** Problem 4.3: Show that the vector potential everywhere due to a charged particle, $e$, moving along the path $r_e(t)$ is

$$\mathbf{A}(x, t) = \frac{e\mathbf{v}_e(\tau)}{c|x - r_e(\tau)| - (x - r_e(\tau)) \cdot \mathbf{v}_e(\tau)}, \quad (4.38)$$

where $\tau$ is the solution to Eq. (4.34).

We proceed to compute the electric and magnetic fields associated with a charged particle, $e$, moving along the path $r_e(t)$. In order to perform the computation, we first note that the expressions for the electromagnetic potential depend on $\tau$ which is a function of $(x, t)$. Differentiating Eq. (4.34) with respect to $t$ at a fixed $x$ gives us

$$\frac{\partial \tau}{\partial t} - 1 - \frac{(x - r_e(\tau)) \cdot \mathbf{v}_e(\tau)}{c|x - r_e(\tau)|} \frac{\partial \tau}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial \tau}{\partial t} = \frac{c|x - r_e(\tau)|}{c|x - r_e(\tau)| - (x - r_e(\tau)) \cdot \mathbf{v}_e(\tau)}. \quad (4.39)$$

Differentiating Eq. (4.34) with respect to one of the components of $x$ keeping the other components and $t$ fixed gives us

$$\frac{\partial \tau}{\partial x} - \frac{(x - r_e(\tau)) \cdot \mathbf{v}_e(\tau)}{c|x - r_e(\tau)|} \frac{\partial \tau}{\partial t} + \frac{x - x_e(\tau)}{c|x - r_e(\tau)|} = 0 \quad \Rightarrow \quad \nabla \tau = -\frac{x - r_e(\tau)}{c|x - r_e(\tau)| - (x - r_e(\tau)) \cdot \mathbf{v}_e(\tau)}. \quad (4.40)$$

The electric field is

$$\mathbf{E}(x, t) = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial \tau} \bigg|_x \frac{\partial \phi}{\partial \tau} - \frac{\partial \phi}{\partial \tau} \bigg|_x \nabla \tau = \frac{e}{c|x - r_e(\tau)| - (x - r_e(\tau)) \cdot \mathbf{v}_e(\tau)} \left[ -\frac{|x - r_e(\tau)|}{c|x - r_e(\tau)| - (x - r_e(\tau)) \cdot \mathbf{v}_e(\tau)} \cdot \mathbf{v}_e(\tau) \right] \cdot \mathbf{a}_e(\tau) + \frac{e}{c|x - r_e(\tau)| - (x - r_e(\tau)) \cdot \mathbf{v}_e(\tau)} \left[ \frac{c(x - r_e(\tau)) \cdot \mathbf{v}_e(\tau)}{|x - r_e(\tau)|} + \mathbf{v}_e(\tau) \cdot \mathbf{a}_e(\tau) - (x - r_e(\tau)) \cdot \mathbf{a}_e(\tau) \right]. \quad (4.41)$$
CHAPTER 4. THE PHYSICS OF INTERACTING CHARGES

4.4 The Darwin Lagrangian

Consider the interaction of a point particle with mass \( m \) and charge \( e \) with a distribution of charges given by \( \rho(x, t) \) and \( j(x, t) \). We will assume that this distribution of charges does not include \( e \) itself. The Lagrangian for the point particle moving along a path \( x(t) \) (after adding a constant) is

\[
L = -mc^2 \left[ \sqrt{1 - \frac{v(t) \cdot v(t)}{c^2}} - 1 \right] - e\phi^0(x(t), t) + \frac{e}{c} A^0(x(t), t) \cdot v(t); \quad v(t) = \frac{dx(t)}{dt},
\]

where \( \phi^0(x, t) \) and \( A^0(x, t) \) are any gauge transformation of \( \phi(x, t) \) and \( A(x, t) \) given by Eq. (4.27). An expansion in powers of \( \frac{1}{c} \) for \( \phi(x, t) \) and \( A(x, t) \) is achieved by perform a Taylor expansion of Eq. (4.27):

\[
\phi(x, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! c^n} \phi^{(n)}(x, t); \quad \phi^{(n)}(x, t) = \frac{\partial^n}{\partial t^n} \left[ \int d^3 x' |x - x'|^{n-1} \rho(x', t) \right];
\]

\[
A(x, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! c^{n+1}} A^{(n)}(x, t); \quad A^{(n)}(x, t) = \frac{\partial^n}{\partial t^n} \left[ \int d^3 x' |x - x'|^{n-1} j(x', t) \right].
\]

We can remove all terms in the expansion for \( \phi(x, t) \) with \( n > 0 \) by performing a gauge transformation,

\[
\phi^0(x, t) = \phi(x, t) - \frac{1}{c} \frac{\partial}{\partial t} \left[ \chi(x, t) = \phi^{(0)}(x, t) \right]; \quad \chi(x, t) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)! c^n} \chi^{(n)}(x, t); \quad A^0(x, t) = A(x, t) + \nabla \left[ \frac{\partial^n}{\partial t^n} \left[ \int d^3 x' |x - x'|^{n-1} \rho(x', t) \right] \right].
\]

We restrict our attention to the case of \( N \) point charges with prescribed paths giving rise to the charge and current distribution. This amounts to evaluating Eq. (4.46) and Eq. (4.47) using Eq. (4.31). The scalar potential is given by

\[
\phi^0(x, t) = \phi^{(0)}(x, t) = \sum_{a=1}^{N} \frac{e_a}{|x - r_a(t)|}.
\]
The leading term for the vector potential is

\[ A(x, t) = \frac{1}{c} A^{(0)}(x, t) + \cdots = \sum_{a=1}^{N} \frac{e_a v_a(t)}{|x - r_a(t)|} + \cdots. \] (4.49)

In order to obtain \( A^g(x, t) \) to leading order, we need to compute

\[ \chi(x, t) = -\chi^{(0)}(x, t) + \frac{1}{2c} \chi^{(1)}(x, t) + \cdots. \] (4.50)

We have

\[ \chi^{(0)}(x, t) = \sum_{a=1}^{N} e_a; \quad \chi^{(1)}(x, t) = \frac{\partial}{\partial t} \left[ \sum_{a=1}^{N} e_a |x - r_a(t)| \right]. \] (4.51)

Therefore,

\[ A^g(x, t) = \sum_{a=1}^{N} \frac{e_a v_a(t)}{|x - r_a(t)|} + \frac{1}{2c} \nabla \left[ \sum_{a=1}^{N} e_a |x - r_a(t)| \right] + \cdots \]
\[ = \sum_{a=1}^{N} \frac{e_a}{c} \left( \frac{v_a(t)}{|x - r_a(t)|} - \frac{1}{2c} \frac{\nabla (x - r_a(t)) \cdot v_a(t)}{|x - r_a(t)|} \right) + \cdots \]
\[ = \sum_{a=1}^{N} \frac{e_a}{2c} \left[ \frac{v_a(t)}{|x - r_a(t)|} + \frac{(x - r_a(t)) (x - r_a(t)) \cdot v_a(t)}{|x - r_a(t)|^3} \right] + \cdots. \] (4.52)

Insertion of Eq. (4.48) and Eq. (4.52) in Eq. (4.45) gives us the Darwin Lagrangian:

\[ L = \frac{1}{2} m v(t) \cdot v(t) + \frac{1}{8c^2} m [v(t) \cdot v(t)]^2 - \sum_{a=1}^{N} \frac{e e_a}{|x - r_a(t)|} \]
\[ + \sum_{a=1}^{N} \frac{e e_a}{2c^2 (x(t) - r_a(t))} \left[ v_a(t) \cdot v(t) + \left\{ \frac{x(t) - r_a(t)}{|x(t) - r_a(t)|} \cdot v_a(t) \right\} \frac{x(t) - r_a(t)}{|x(t) - r_a(t)|} \right]. \] (4.53)

**Problem 4.10:** Derive the equations of motion for the charged particle of mass \( m \) and \( e \) due to its interaction with \( N \) other charges. You have to work this problem out in two ways and show that you get the same result. For the first method, use Eq. (4.53). For the second method, compute the electric field and magnetic field using Eq. (4.48) and Eq. (4.52) and then use the Lorentz force.

**Points Problem 4.11:** Derive the Lagrangian equivalent to the one in Eq. (4.53) obtained by using the advanced Green’s function given in Eq. (4.25). Is there any difference? If so, explain why it makes physical sense (or not).

**Points Problem 4.12:** Starting from Eq. (4.53), derive the Lagrangian to the same order for \( N \) charged particles with charge \( e_a \) and mass \( m_a \) \( (a = 1, \cdots, N) \) interacting with each other. You can ignore the self interaction.

**Points Problem 4.13:** Extend Eq. (4.53) to include terms up to \( \frac{1}{c^4} \). Derive the equations of motion for the charged particle of mass \( m \) and \( e \). Discuss the physics in detail. You do not have to include self interaction. You may find the following reference useful: B.M. Barker and R.F. O’Connell, Annals of Physics, 129, 358-377 (1980). You may obtain a pdf copy of the paper from the following Louisiana State University website (Paper number 103 in http://phys.lsu.edu/graceland/faculty/occonnell/occonnell_pubs.html). Reference 11 in this paper, P.A.M. Dirac, Proc. Roy. Soc. Sect. A 167 (1938) 148, is also a useful paper to work out this problem.
Chapter 5

Radiation

5.1 Emergent plane waves

Consider a charge and current distribution that is restricted to a small volume centered around the origin of our coordinate system for the entire time. In this chapter, we will be interested in studying the properties of electromagnetic fields due to this confined system at asymptotic distances away from the system. Our starting point will be the retarded potentials given by Eq. (4.27):

\[ \phi(x, t) = \int d^3x' dt' \frac{\rho(x', t')}{|x - x'|} \delta \left( t' - t + \frac{|x - x'|}{c} \right) ; \quad \mathbf{A}(x, t) = \int d^3x' dt' \frac{j(x', t')}{c|x - x'|} \delta \left( t' - t + \frac{|x - x'|}{c} \right). \]

Anticipating the wave nature of the solutions at asymptotic distances, we perform a Fourier transform of the potentials:

\[ \tilde{\phi}(x, \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \phi(x, t); \quad \tilde{\mathbf{A}}(x, \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \mathbf{A}(x, t). \] (5.1)

The associated inverse Fourier transforms are

\[ \phi(x, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\phi}(x, \omega); \quad \mathbf{A}(x, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\mathbf{A}}(x, \omega), \] (5.2)

obtained by using the definition of the delta function in Eq. (4.13). Inserting Eq. (5.1) into the expressions for the retarded potentials and performing the integral over \( t \), we obtain

\[ \tilde{\phi}(x, \omega) = \int d^3x' dt' \frac{\tilde{\rho}(x', \omega)}{|x - x'|} e^{i\omega (t' + \frac{|x - x'|}{c})}; \quad \tilde{\mathbf{A}}(x, \omega) = \int d^3x' dt' \frac{\tilde{j}(x', \omega)}{c|x - x'|} e^{i\omega (t' + \frac{|x - x'|}{c})}. \] (5.3)

Let us define the magnitude and direction of \( x \) as

\[ |x| = r; \quad \frac{x}{r} = \mathbf{n}. \] (5.4)

Then,

\[ |x - x'| = r - \mathbf{n} \cdot \mathbf{x'} + O \left( \frac{1}{r} \right), \] (5.5)

and

\[ \tilde{\phi}(x, \omega) = \frac{e^{ikr}}{r} \int d^3x' \tilde{\rho}(x', \omega)e^{-ik\mathbf{n} \cdot \mathbf{x'}} + O \left( \frac{1}{r} \right); \quad \tilde{\mathbf{A}}(x, \omega) = \frac{e^{ikr}}{cr} \int d^3x' \tilde{j}(x', \omega)e^{-ik\mathbf{n} \cdot \mathbf{x'}} + O \left( \frac{1}{r} \right). \] (5.6)

In terms of moments,

\[ \tilde{\phi}(x, \omega) = \frac{e^{ikr}}{r} \sum_{m=0}^{\infty} (-ik)^m \tilde{\rho}^{(m)}(\omega, \mathbf{n}); \quad \tilde{\rho}^{(m)}(\omega, \mathbf{n}) = \frac{1}{m!} \int d^3x' \tilde{\rho}(x', \omega)(\mathbf{n} \cdot \mathbf{x'})^m; \]

\[ \tilde{\mathbf{A}}(x, \omega) = \frac{e^{ikr}}{cr} \sum_{m=0}^{\infty} (-ik)^m \tilde{\mathbf{A}}^{(m)}(\omega, \mathbf{n}); \quad \tilde{\mathbf{A}}^{(m)}(\omega, \mathbf{n}) = \frac{1}{m!} \int d^3x' \tilde{j}(x', \omega)(\mathbf{n} \cdot \mathbf{x'})^m. \] (5.7)
We revert back to functions of time using Eq. (5.2) and obtain

\[
\phi(x,t) = \frac{1}{r} \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( -\frac{i\omega}{c} \right)^m \tilde{\rho}^{(m)}(\omega, n)e^{-i\omega(t-\tau)}
\]

\[
= \frac{1}{r} \sum_{m=0}^{\infty} \left( \frac{\partial}{\partial t} \right)_{x} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\rho}^{(m)}(\omega, n)e^{-i\omega(t-\tau)}
\]

\[
= \frac{\tilde{\phi}(\tau, n)}{r}, \quad \tau = t - \frac{r}{c}, \quad (5.8)
\]

where

\[
\tilde{\phi}(\tau, n) = \sum_{m=0}^{\infty} \tilde{\phi}^{(m)}(\tau, n); \quad \tilde{\phi}^{(m)}(\tau, n) = \left( \frac{\partial}{\partial \tau} \right)_{x} \rho^{(m)}(\tau, n); \quad \rho^{(m)}(\tau, n) = \frac{1}{m!} \int d^3 x' \rho(x', \tau) (n \cdot x')^m.
\]

In a similar manner,

\[
A(x,t) = \tilde{A}(\tau, n) c r, \quad \tilde{A}(\tau, n) = \sum_{m=0}^{\infty} \tilde{A}^{(m)}(\tau, n);
\]

\[
\tilde{A}^{(m)}(\tau, n) = \left( \frac{\partial}{\partial \tau} \right)_{x} \tilde{j}^{(m)}(\tau, n); \quad \tilde{j}^{(m)}(\tau, n) = \frac{1}{m!} \int d^3 x' \tilde{j}(x', \tau) (n \cdot x')^m.
\]

In order to compute the electric and magnetic fields to leading order in \(1/p\), we note that

- A derivative with respect to \(t\) at fixed \(r\) only acts on \(\tau\) and

\[
\left. \frac{\partial \tau}{\partial t} \right|_r = 1. \quad (5.11)
\]

- A derivative of \(\tau\) with respect to \(r\) at fixed \(t\) is

\[
\nabla \tau = -\frac{n}{c}, \quad (5.12)
\]

and is relevant at leading order in \(1/p\).

- A derivative of \(\frac{1}{r}\) with respect to \(r\) at fixed \(t\) is

\[
\nabla \frac{1}{r} = -\frac{n}{r^2}, \quad (5.13)
\]

and is not relevant at leading order in \(1/p\).

- A derivative of \(n_i\) with respect to \(x_j\) at fixed \(t\) is

\[
\left. \frac{\partial n_i}{\partial x_j} \right|_r = \frac{\delta_{ij}}{r} - \frac{x_j x_i}{r^3}, \quad (5.14)
\]

and therefore is not relevant at leading order in \(1/p\).

All of the above statements imply that

\[
\left[ n \cdot \nabla + \frac{1}{c} \frac{\partial}{\partial t} \right] \left( \frac{\phi(x,t)}{A(x,t)} \right) = 0, \quad (5.15)
\]

to leading order in \(1/p\), which is a solution to the wave equation in Eq. (5.39).

The magnetic field to leading order in \(1/p\) is

\[
H(x,t) = \nabla \times A(x,t) = \frac{1}{cr} (\nabla \tau) \times \left( \frac{\partial \tilde{A}(\tau, n)}{\partial \tau} \right) = \frac{1}{c^2 r} \frac{\partial \tilde{A}(\tau, n)}{\partial \tau} \times n. \quad (5.16)
\]

The electric field to leading order in \(1/p\) is

\[
E(x,t) = -\nabla \phi(x,t) - \frac{1}{c} \frac{\partial A(x,t)}{\partial t} = \frac{1}{c r} \frac{\partial \tilde{\phi}(\tau, n)}{\partial \tau} n - \frac{1}{c^2 r} \frac{\partial \tilde{A}(\tau, n)}{\partial \tau}. \quad (5.17)
\]
• **points: Problem 5.1:** Show that
\[ \mathbf{n} \times \mathbf{E}(\mathbf{x}, t) = \mathbf{H}(\mathbf{x}, t). \] (5.18)

• **points: Problem 5.2:** Show that
\[ \mathbf{n} \cdot \mathbf{E}(\mathbf{x}, t) = 0. \] (5.19)

• **points: Problem 5.3:** Show that
\[ \mathbf{H}(\mathbf{x}, t) \times \mathbf{n} = \mathbf{E}(\mathbf{x}, t). \] (5.20)

These properties of the electric and magnetic field show that they are indeed transverse electromagnetic waves to leading order in \( \frac{1}{r} \). Since the energy flux
\[ \mathbf{S}(\mathbf{x}, t) = \frac{c}{4\pi} \mathbf{E}(\mathbf{x}, t) \times \mathbf{H}(\mathbf{x}, t), \] (5.21)
is proportional to \( \frac{1}{r^2} \), this energy loss is finite when integrated over a very large sphere of radius \( r \) enclosing the charges and is the energy being radiated by the system of charges. For calculational purposes, it is sufficient to compute the vector potential \( \mathbf{A}(\mathbf{x}, t) \) and use it to compute the magnetic field. The electric field can then be computed using Eq. (5.20).

### 5.2 Electric dipole radiation

We will show that the \( m = 0 \) term in Eq. (5.10) arises from an electric dipole term. For the case of \( m = 0 \), we have
\[ \mathbf{A}^{(0)}(\tau, \mathbf{n}) = \mathbf{j}^{(0)}(\tau, \mathbf{n}) = \int d^3x' \mathbf{j}(\mathbf{x}', \tau), \] (5.22)
and the result is independent of \( \mathbf{n} \). The integral can be written in terms of the dipole moment. In order to show this we proceed as follows. First of all, we assume that the volume integral over \( x' \) is such that there is no current or charge on the boundary at any time. Therefore, we can write
\[ 0 = \int d^3x' \sum_k \frac{\partial [j'_k(x', \tau)]}{\partial x'_k} = \int d^3x' \left[ \sum_i \frac{\partial j_k(x', \tau)}{\partial x'_k} + \sum_k \delta_{ik} j_k(x', \tau) \right]. \] (5.23)

In vector notation, we have the identity,
\[ \int d^3x' \mathbf{j}(\mathbf{x}', \tau) = - \int d^3x' \mathbf{x}' \left[ \nabla' \cdot \mathbf{j}(\mathbf{x}', \tau) \right]. \] (5.24)

We can use the local conservation of charge at any time, \( \tau \),
\[ \nabla' \cdot \mathbf{j}(\mathbf{x}', \tau) + \frac{\partial \rho(\mathbf{x}', \tau)}{\partial \tau} = 0; \] (5.25)
to rewrite the identity in terms of the dipole moment, \( \mathbf{d}(\tau) \), as
\[ \mathbf{A}^{(0)}(\tau, \mathbf{n}) = \int d^3x' \mathbf{j}(\mathbf{x}', \tau) = \int d^3x' \mathbf{x}' \frac{\partial \rho(\mathbf{x}', \tau)}{\partial \tau} = \frac{d}{d\tau} \int d^3x' \rho(\mathbf{x}', \tau) = \frac{d\mathbf{d}(\tau)}{d\tau}. \] (5.26)

For the case of a set of discrete charges given by Eq. (4.31),
\[ \mathbf{A}^{(0)}(\tau, \mathbf{n}) = \frac{d}{d\tau} \left[ \sum_{a=1}^{N} e_a \mathbf{r}_a(\tau) \right] = \sum_{a=1}^{N} e_a \mathbf{v}_a(\tau). \] (5.27)

The vector potential associated with the electric dipole radiation is
\[ \mathbf{A}_{ed}(\mathbf{x}, \tau) = \sum_{a=1}^{N} e_a \mathbf{v}_a(\tau) \left( t - \frac{r}{c} \right). \] (5.28)

The associated magnetic field given by Eq. (5.16) is
\[ \mathbf{H}_{ed}(\mathbf{x}, \tau) = \frac{1}{c^2r} \frac{d^2\mathbf{d}(\tau)}{d\tau^2} \times \mathbf{n} = \sum_{a=1}^{N} e_a \mathbf{a}_a(\tau) \left( t - \frac{r}{c} \right) \times \mathbf{n}. \] (5.29)
CHAPTER 5. RADIATION

In vector notation, we have the identity
\[ m = \text{ term in Eq. (5.10)} \]
evaluated at \( \tau = t - \frac{r}{c} \). Hence show that for a single charge \( e \) moving along a confined path \( r(t) \) over a sphere of very large radius \( r \) enclosing the charge at all times, the total power is given by
\[ I_{ed}(r,t) = \frac{2e^2a^2(t - \frac{r}{c})}{3c^3}, \]
where \( a(\tau) \) is the magnitude of the acceleration of the charge at time \( \tau \).

- **Problem 5.4:** Show that there can be no dipole radiation from a closed system of charged particles where all particles have the same charge to mass ratio. A closed system, by definition, does not have any external influence.

- **Problem 5.5:** Show that there can be no dipole radiation from a closed system of charged particles where all particles have the same charge to mass ratio. A closed system, by definition, does not have any external influence.

### 5.3 Magnetic dipole and electric quadrupole radiation

We will show that a magnetic dipole and an electric quadrupole contribute to the \( m = 1 \) term in Eq. (5.10). For the case of \( m = 1 \), we have
\[ \mathbf{A}^{(1)}(\tau, \mathbf{n}) = \frac{\partial}{\partial \tau} \left[ \mathbf{j}^{(1)}(\tau, \mathbf{n}) \right] ; \quad \mathbf{j}^{(1)}(\tau, \mathbf{n}) = \int d^3x' \mathbf{j}(x', \tau) \cdot [\mathbf{n} \cdot \mathbf{x}']. \]

In order to see the presence of a magnetic dipole in the above equation we write
\[
\int d^3x' \mathbf{j}(x', \tau) \cdot [\mathbf{n} \cdot \mathbf{x}'] = \frac{1}{2} \int d^3x' \left[ [\mathbf{x} \times \mathbf{j}(x', \tau)] \cdot \mathbf{n} \right] \cdot [\mathbf{x} \cdot \mathbf{n}] + \frac{1}{2} \int d^3x' \mathbf{j}(x', \tau) \cdot [\mathbf{n} \cdot \mathbf{x}] + \mathbf{x} \cdot [\mathbf{n} \cdot \mathbf{j}(x', \tau)]
\]
\[
= c \mathbf{m}(\tau) \times \mathbf{n} + \frac{1}{2} \int d^3x' \mathbf{j}(x', \tau) \cdot [\mathbf{n} \cdot \mathbf{x}] + \mathbf{x} \cdot [\mathbf{n} \cdot \mathbf{j}(x', \tau)]
\]
and we see that the first term corresponds to a magnetic dipole,
\[ \mathbf{m}(\tau) = \frac{1}{2c} \int d^3x' \left[ \mathbf{x} \times \mathbf{j}(x', \tau) \right]. \]

In order to see the presence of an electric quadrupole in the second term in Eq. (5.33), we start by writing
\[ 0 = \int d^3x' \left( \sum_{k,l} \frac{\partial \left[ \mathbf{x}_k^{l} \mathbf{j}(x', \tau) \right]}{\partial x'_l} n_k \right)
\]
\[ = \int d^3x' \left( \sum_{k} \mathbf{x}_k n_k \right) \sum l \frac{\partial j_l(x', \tau)}{\partial x'_l} + \sum l \left( \sum k n_k (\delta_{il} x'_k + \delta_{kl} x'_l) \right) j_l(x', \tau) \right]. \]

In vector notation, we have the identity
\[
\int d^3x' \left[ [\mathbf{x} \times \mathbf{j}(x', \tau)] \cdot \mathbf{n} \right] \cdot [\mathbf{x} \cdot \mathbf{n}]
\]
\[ = - \int d^3x' \cdot [\mathbf{x} \times \mathbf{n}] \mathbf{j}(x', \tau)
\]
\[ = \int d^3x' \cdot [\mathbf{x} \times \mathbf{n}] \frac{\partial \rho(x', \tau)}{\partial \tau}
\]
\[ = \frac{d}{dt} \int d^3x' \cdot [\mathbf{x} \times \mathbf{n}] \rho(x', \tau)
\]
\[ = \frac{1}{3} \frac{d}{dt} \int d^3x' \left[ 3\mathbf{x} \cdot \mathbf{n} - (\mathbf{x} \cdot \mathbf{n}) \mathbf{n} \right] \rho(x', \tau) + \frac{n}{3} \frac{d}{dt} \int d^3x' \cdot [\mathbf{x} \times \mathbf{x}] \rho(x', \tau) \]
5.4. RADIATION IN THE CASE OF A COULOMB INTERACTION

5.4.1 Classical non-relativistic motion

\[ \frac{1}{3} \frac{dQ(\tau)}{d\tau} \cdot \mathbf{n} + \frac{n}{3} \frac{d}{d\tau} \int d^3x' (x' \cdot x') \rho(x', \tau), \]  

(5.36)

where

\[ Q_{ij}(\tau) = \int d^3x' \left[ 3x'_i x'_j - x'_i x'_j \delta_{ij} \right] \rho(x', \tau) \]

(5.37)
is the traceless electric quadrupole tensor and

\[ [\mathbf{Q} \cdot \mathbf{n}]_i = \sum_j Q_{ij} n_j. \]

(5.38)

Inserting Eq. (5.36) into Eq. (5.33) and the result into Eq. (5.32), we obtain

\[ \mathbf{A}^{(1)}(\tau, \mathbf{n}) = \frac{d\mathbf{m}(\tau)}{d\tau} \times \mathbf{n} + \frac{1}{6c} \frac{d^2 \mathbf{Q}(\tau)}{d\tau^2} \cdot \mathbf{n} + \frac{n}{6c^2} \frac{d^3x'}{d\tau^3} (x' \cdot x') \rho(x', \tau). \]

(5.39)
The first term is the contribution from the magnetic dipole and the second term is from the electric quadrupole. The third term is pointing in the direction of \( \mathbf{n} \) and therefore will not contribute to the magnetic field or electric field. The associated magnetic fields from the magnetic dipole and the electric quadrupole are

\[ \mathbf{H}_{\text{md}} = \frac{1}{c^2 r} \left\{ \frac{d^2 \mathbf{m}(\tau)}{d\tau^2} \times \mathbf{n} \right\} \times \mathbf{n}; \quad \mathbf{H}_{\text{eq}} = \frac{1}{6c^3 r} \left\{ \frac{d^3 \mathbf{Q}(\tau)}{d\tau^3} \cdot \mathbf{n} \right\} \times \mathbf{n}, \]

(5.40)

respectively.

- **points**: Problem 5.6: Consider a very simple model for a proton where it is made up of two \( u \) quarks and one \( d \) quark. The \( u \) quark has a charge of \( \frac{2}{3} \) and the \( d \) quark has a charge of \( -\frac{1}{3} \) in units of the charge of a positron. All quarks move in a circle of radius \( r \) in the \( x - y \) plane centered at \( x = y = z = 0 \) with a constant angular velocity of \( \omega \). The three charges are separated from each other by \( \frac{2\pi}{\tau} \) on the circle of radius \( r \) at \( t = 0 \) and all of them have a counter-clockwise motion. Derive the radiation fields (both electric and magnetic) and separate the contributions (if any) from the electric dipole, magnetic dipole and electric quadrupole. Compute the power radiated by the proton. Assuming that the effect of radiation is to reduce \( \omega \) at a fixed \( r \), how long will it take for the proton to lose all its initial energy? You can assume both \( u \) and \( d \) quarks have the same mass \( m \) and also assume that \( r \) and \( \omega \) are such that you can use the non-relativistic formula for the kinetic energy. How will all your answers change if the \( u \) quarks were moving counter-clockwise but the \( d \) quark was moving clockwise.

- **points**: Problem 5.7: Repeat all questions in the previous problem for a neutron which is made up of two \( d \) quarks and one \( u \) quark.

5.4 Radiation in the case of a Coulomb interaction

Consider a particle with charge \( e_1 \) and mass \( m_1 \) interacting with another particle with charge \( e_2 \) and mass \( m_2 \). The charges could either repel or attract each other. Assuming that the velocities of the particles are small, we will assume non-relativistic mechanics holds and solve for the motion of the two particles. We will use this result to derive the radiation from these two charges.

5.4.1 Classical non-relativistic motion

- **points**: Problem 5.8: Let \( \mathbf{r}_1(t) \) and \( \mathbf{r}_2(t) \) define the motion of the two particles. Write down the equation of motion for the two particles assuming that the only force is the instantaneous Coulomb force between the two particles. Then show that one can always go to an inertial frame whose origin is defined by

\[ m_1 \mathbf{r}_1(t) + m_2 \mathbf{r}_2(t) = 0. \]

(5.41)

Define

\[ \mathbf{r}_p = \mathbf{r}_1 - \mathbf{r}_2. \]

(5.42)

Show that the only equation of motion one needs to solve is

\[ m \frac{d^2 \mathbf{r}_p(t)}{dt^2} = \frac{e_1 e_2}{r_p^2} \mathbf{r}_p; \quad m = \frac{m_1 m_2}{m_1 + m_2}; \quad \mathbf{r}_p = |\mathbf{r}_p|; \quad \dot{r}_p = \frac{\mathbf{r}_p}{r_p}, \]

(5.43)
• – points: Problem 5.9: Show that the angular momentum,
\[ L = m r_p(t) \times v_p(t); \quad v_p(t) = \frac{dr_p(t)}{dt}, \]
(5.44)
is conserved.

• – points: Problem 5.10: Since the angular momentum is conserved we can assume \( L = L \hat{z} \). Show that the motion is restricted to the \( x - y \) plane.

• – points: Problem 5.11: Use polar coordinates, namely,
\[ x_p(t) = r_p(t) \cos \phi(t); \quad y_p(t) = r_p(t) \sin \phi(t); \]
(5.45)
and show that
\[ L = m r_p^2 \frac{d\phi}{dt}. \]
(5.46)

• – points: Problem 5.12: Show that the total energy
\[ E = \frac{1}{2} m \left[ \left( \frac{dr_p}{dt} \right)^2 + r_p^2 \left( \frac{d\phi}{dt} \right)^2 \right] + \frac{e_1 e_2}{r_p}, \]
(5.47)
is conserved.

• – points: Problem 5.13: Show that the electric dipole moment can be written as
\[ d(t) = m \left( \frac{e_1}{m_1} - \frac{e_2}{m_2} \right) r_p(t). \]
(5.48)
Let us focus on the attractive case and set
\[ \alpha = -e_1 e_2 > 0. \]
(5.49)
We can write
\[ \frac{dr_p}{dt} = r_p' \frac{d\phi}{dt}; \quad r_p' = \frac{dr_p}{d\phi}, \]
(5.50)
Inserting Eq. (5.50) into Eq. (5.47) and substituting for \( \frac{d\phi}{dt} \) using Eq. (5.46), we obtain
\[ E = \frac{1}{2} m \left( r_p^2 + r_p^2 \right) \frac{L^2}{m^2 r_p^4} - \frac{\alpha}{r_p}. \]
(5.51)
We will also assume that the orbit is bound and set \( E < 0 \). Let us define
\[ u = \frac{1}{r_p}; \quad r_p' = -\frac{u'}{u^2}, \]
(5.52)
and Eq. (5.51) becomes
\[ u'^2 + u^2 - \frac{2m\alpha}{L^2} u = \frac{2mE}{L^2} \Rightarrow u'^2 + \left( u - \frac{m\alpha}{L^2} \right)^2 = \frac{2mE}{L^2} + \frac{m^2 \alpha^2}{L^4}. \]
(5.53)
This gives us the minimum total energy for a closed orbit, namely,
\[ E > E_{\text{min}} = -\frac{m\alpha^2}{2L^2}. \]
(5.54)
We define
\[ p = \frac{L^2}{m\alpha}; \]
(5.55)
and perform another change of variable,
\[ z = pu - 1. \]
(5.56)
This reduces Eq. (5.53) to
\[ z'^2 + z^2 = 1 + \frac{2mEp^2}{L^2} = 1 + \frac{2L^2E}{m\alpha^2} = 1 + \frac{2E}{\alpha} = \epsilon^2. \] (5.57)
The solution to the above equation with appropriately chosen initial conditions is
\[ z = \epsilon \cos \phi. \] (5.58)
Given \( \alpha \), our two constants of motions are \( E \) and \( L \). Instead of \( E \) we will use
\[ a = -\frac{\alpha}{2E} > 0 \] (5.59)
as one of our constants of motion. We will set \( \epsilon \) as the other constant of motion and write
\[ \frac{L^2}{m\alpha} = p = (1 - \epsilon^2)a \quad \Rightarrow \quad L = \sqrt{ma}(1 - \epsilon^2). \] (5.60)

• – points: Problem 5.14: Derive the units of \( a \) and \( \epsilon \)? Derive the range of \( \epsilon \) for bound orbits.

The solution in Eq. (5.58) can be written using Eq. (5.52), Eq. (5.56) and Eq. (5.60) as
\[ (1 - \epsilon^2)a = \frac{rp}{r^p} = 1 + \epsilon \cos \phi. \] (5.61)
At this point we can insert Eq. (5.61) into Eq. (5.46) and solve for \( \phi(t) \). It turns out this is not convenient for the analysis we wish to do for radiation. With this in mind, we go back to Eq. (5.47) and insert Eq. (5.46) to obtain
\[ \frac{1}{2m} \left[ \frac{d(r^p)}{dt} \right]^2 + \frac{L^2}{m^2r^p} - \frac{\alpha}{r^p} = E; \]
\[ \left( \frac{dr^p}{dt} \right)^2 = \frac{2E}{m} + \frac{2\alpha}{mr^p} - \frac{L^2}{m^2r^p}; \]
\[ = -\frac{2E}{mr^p} \left( -r^2 + \frac{\alpha}{E}r + \frac{L^2}{2mE} \right) \]
\[ = \frac{\alpha}{amrp^2} \left( -r^2 + 2ar - pa \right) \]
\[ = \frac{\alpha}{amrp^2} \left( -[r^p - a]^2 + a^2 \left( 1 - \frac{p}{a} \right) \right) \]
\[ = \frac{\alpha}{amrp^2} \left( -[r^p - a]^2 + a^2 \epsilon^2 \right) \] (5.62)
We have used Eq. (5.59) and Eq. (5.55) in the fourth line to replace \( E \) and \( L \) in terms of \( a \) and \( p \). We have used Eq. (5.60) in the last line. We define an auxiliary variable, \( \xi(t) \), such that
\[ r^p - a = -a\epsilon \cos \xi. \] (5.63)
Inserting this into Eq. (5.62) results in
\[ a^2 \epsilon^2 \sin^2 \xi \left( \frac{d\xi}{dt} \right)^2 = \frac{\alpha}{amrp^2} a^2 \epsilon^2 \sin^2 \xi \quad \Rightarrow \quad \left( \frac{d\xi}{dt} \right)^2 = \frac{\alpha}{a^3m(1 - \epsilon \cos \xi)^2} \quad \Rightarrow \quad \frac{dt}{d\xi} = \sqrt{\frac{a^3m}{\alpha}} \frac{1 - \epsilon \cos \xi}{\alpha}. \] (5.64)
The solution for \( t \) in terms of our auxiliary variable is
\[ t = \sqrt{\frac{a^3m}{\alpha}} (\xi - \epsilon \sin \xi) \quad \Rightarrow \quad \omega_0 t = \xi - \epsilon \sin \xi; \quad \omega_0 = \sqrt{\frac{\alpha}{a^3m}}; \quad T_0 = \frac{2\pi}{\omega_0}. \] (5.65)

• – points: Problem 5.15: Prove that \( t \) is a monotonically increasing function of \( \xi \).

We can use Eq. (5.61) and write
\[ (1 - \epsilon^2)a = r^p + \epsilon r_p \cos \phi = r^p + \epsilon x_p. \] (5.66)
Using Eq. (5.63), we obtain
\[ \epsilon x_p = (1 - \epsilon^2)a - a + a \epsilon \cos \xi = a \epsilon ( \cos \xi - \epsilon ); \quad \Rightarrow \quad x_p = a ( \cos \xi - \epsilon ). \quad (5.67) \]

Using Eq. (5.63) and Eq. (5.67), we have
\[ y_p = \sqrt{r_p^2 - x_p^2} = a \sqrt{(1 - \epsilon \cos \xi)^2 - (\cos \xi - \epsilon)^2} = a \sqrt{1 - \epsilon^2} \sin \xi. \quad (5.68) \]

Note that \( r_p \) are periodic functions of \( \xi \). Furthermore, Eq. (5.65) tells us that a total time of \( T_0 \) is elapsed as \( \xi \) completes one period of length \( 2\pi \).

### 5.4.2 Dipole radiation

- **Problem 5.16:** Using the formula for the total power due to dipole radiation given in Eq. (5.30) and using the equation of motion in Eq. (5.43) prove the average power radiated in the period \( T_0 \) defined in Eq. (5.65) is
\[ I_{av} = \sqrt{-8E^3 \frac{\alpha^3 m^3}{3e^3 L^5}} \left( e_1 \frac{m}{m_1} - e_2 \frac{m}{m_2} \right)^2 \left( 3 + 2EL^2 \frac{m \alpha^2}{m^3} \right). \quad (5.69) \]

You may find it useful to convert the integral over \( t \) to an integral over \( \phi \) using Eq. (5.45). Use this formula to compute the average power lost in one period by a Hydrogen atom assuming a classical circular orbit.

There is more information in the dipole radiation since the particle motion is not a simple motion and its decomposition into frequencies will include all harmonics of \( \omega_0 \). In order to obtain this information, we perform a Fourier decomposition of \( r_p \). In order to eliminate zero frequency modes, we shift the origin along the \( x \)-axis by \( a \epsilon \). We can write
\[ r_p(t) + a \epsilon = \sum_{n=-\infty}^{\infty} r_n e^{-i\omega_0 nt}; \quad r_{-n} = r_n^*; \quad r_0 = 0. \quad (5.70) \]

Inserting this expression into the total power from the dipole radiation given in Eq. (5.30) after using the expression for the dipole in Eq. (5.48) we obtain
\[ I(r, t) = \frac{2n^2 \omega_0^4}{3e^3} \left( \frac{e_1}{m_1} - \frac{e_2}{m_2} \right)^2 \left| \sum_{n=-\infty}^{\infty} n^2 r_n e^{-i\omega_0 n(t - \xi)} \right|^2. \quad (5.71) \]

Averaging over one period, \( T_0 \), we obtain
\[ I_{av} = \sum_{k=1}^{\infty} I_k; \quad I_k = \frac{4n^2 \omega_0^4 k^4}{3e^3} \left( \frac{e_1}{m_1} - \frac{e_2}{m_2} \right)^2 (r_k \cdot r_k^*). \quad (5.72) \]

Differentiating with respect to \( t \) gives us
\[ \frac{dr_p(t)}{dt} = - \sum_{n=-\infty}^{\infty} i \omega_0 n r_n e^{-i\omega_0 nt} \Rightarrow \frac{1}{T_0} \int_0^{T_0} \frac{dr_p(t)}{dt} \frac{dt}{dt} e^{i\omega_0 k t} dt = -i \omega_0 k r_k. \quad (5.73) \]

Therefore we can obtain all the Fourier components from
\[ r_k = \frac{i}{2\pi k} \int_0^{T_0} \frac{dr_p(t)}{dt} e^{i\omega_0 k t} dt = \frac{i}{2\pi k} \int_0^{2\pi} \frac{dr_p(\xi)}{d\xi} e^{ik(\xi - \epsilon \sin \xi)} d\xi, \quad (5.74) \]

where we have used Eq. (5.65) to change integration variables from \( t \) to \( \xi \). We now use Eq. (5.67) and Eq. (5.68) to write the above equation in component form as
\[ x_k = \frac{ia}{2\pi k} \int_0^{2\pi} \sin \xi e^{ik(\xi - \epsilon \sin \xi)} d\xi; \quad y_k = \frac{ia}{2\pi k} \int_0^{2\pi} \cos \xi e^{ik(\xi - \epsilon \sin \xi)} d\xi. \quad (5.75) \]

We can evaluate \( x_k \) and \( y_k \) in terms of the Bessel functions,
\[ J_k(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} e^{ikx - i\alpha \sin \xi} d\xi. \quad (5.76) \]
5.5. RADIATION FROM A SINGLE MOVING CHARGE

Differentiating with respect to \( \alpha \) gives us

\[
x_k = \frac{a}{k} \frac{dJ_k(\alpha)}{d\alpha} \bigg|_{\alpha = k \epsilon}.
\]

(5.77)

Noting that

\[
0 = \int_0^{2\pi} d \left( e^{ik\xi - ike\sin \xi} \right) = ik \int_0^{2\pi} (1 - e\cos \xi) e^{ik\xi - ike\sin \xi} d\xi,
\]

we obtain

\[
y_k = \frac{ia\sqrt{1 - \epsilon^2}}{k \epsilon} J_k(k \epsilon).
\]

(5.79)

Inserting our results for \( x_k \) and \( y_k \) in Eq. (5.72), we obtain

\[
I_k = \frac{4m^2a^2\omega_0^2k^2}{3c^3} \left( \frac{e_1}{m_1} - \frac{e_2}{m_2} \right)^2 \left( \left[ \frac{dJ_k(\alpha)}{d\alpha} \right]_{\alpha = k \epsilon} \right)^2 + \frac{1 - \epsilon^2}{\epsilon^2} J_k^2(k \epsilon). \]

(5.80)

We have used Eq. (5.59) and Eq. (5.65) to obtain the last line.

- **points: Problem 5.17:** Following a procedure similar to the one for closed orbits, obtain the frequency decomposition of the total energy radiated due to dipole radiation during the collision of two attracting particles. You will need to use a continuous Fourier transform instead of a discrete Fourier series.

- **points: Problem 5.18:** Following a procedure similar to the one for closed orbits, obtain the frequency decomposition of the total energy radiated due to dipole radiation during the collision of two repelling particles. You will need to use a continuous Fourier transform instead of a discrete Fourier series.

5.5 Radiation from a single moving charge

The Lienard-Wiechert expressions for the electric and magnetic field in Eq. (4.43) and Eq. (4.44) can be used to compute the radiation due to a single moving charge. In order to compute the radiated power we can assume that we are far away from the moving charge which is assumed to be confined to a finite region around the origin for the entire time period. As before, we expand

\[
|x - r_e(\tau)| = r - n \cdot r_e(\tau) + O \left( \frac{1}{r} \right).
\]

(5.81)

We can then write

\[
g(x, \tau) = (c - n \cdot v_e(\tau)) r - (c n - v_e(\tau)) \cdot r_e(\tau) + O \left( \frac{1}{r} \right),
\]

(5.82)

and

\[
f(x, \tau) = (c n - v_e(\tau)) r - cr_e(\tau) + (n \cdot r_e(\tau)) v_e(\tau) + O \left( \frac{1}{r} \right).
\]

(5.83)

Inserting these two expressions, we obtain the following expansion for the electric field:

\[
E(x, t) = \frac{c}{r} \frac{n \times \{(c n - v_e(\tau)) \times a_e(\tau)\}}{(c - n \cdot v_e(\tau))^3} + O \left( \frac{1}{r^2} \right).
\]

(5.84)

We obtain the following expression for the magnetic field:

\[
H(x, t) = n \times E(x, t) + O \left( \frac{1}{r^2} \right).
\]

(5.85)

The power radiated in a solid angle \( d\Omega \) is

\[
\frac{dS}{d\Omega} = \frac{c}{4\pi} E^2 r^2
\]

(5.86)
which can be simplified to
\[
\frac{dS}{d\Omega} = \frac{e^2}{4\pi c^3} \left| \frac{n}{c} \times \left( \frac{n - \nu_s(\tau)}{c} \times a_s(\tau) \right) \right|^2 \left( \frac{1 - \nu_s(\tau)}{c} \right)^6
\]
\[
= \frac{e^2}{4\pi c^3} \left| \left( \frac{n - \nu_s(\tau)}{c} \right) (n \cdot a_s(\tau)) - a_s(\tau) \left( 1 - \frac{n \nu_s(\tau)}{c} \right)^2 \right|^2 \left( \frac{1 - \nu_s(\tau)}{c} \right)^6
\]
\[
= \frac{e^2}{4\pi c^3} \left[ \frac{a_s^2(\tau)}{\left( 1 - \frac{n \nu_s(\tau)}{c} \right)^4} + \frac{2 (n \cdot a_s(\tau)) \left( \nu_s(\tau) \cdot a_s(\tau) \right)}{c \left( 1 - \frac{n \nu_s(\tau)}{c} \right)^4} - \frac{\left( 1 - \frac{\nu_s^2(\tau)}{c^2} \right) (n \cdot a_s(\tau))^2}{\left( 1 - \frac{n \nu_s(\tau)}{c} \right)^6} \right].
\]
(5.87)

Noting that \( S \) is the power radiated, we can write
\[
\frac{dS}{d\Omega} = \frac{d\mathcal{E}}{dt}\frac{1}{dt}.
\]
(5.88)

where \( \mathcal{E} \) is the radiated energy. Using Eq. (4.39), we have
\[
\frac{d\tau}{dt} = \frac{1}{1 - \frac{n \nu_s(\tau)}{c}} + O \left( \frac{1}{t^2} \right),
\]
(5.89)
and therefore
\[
\frac{d\mathcal{E}}{dt} = \frac{e^2}{4\pi c^3} \left[ \frac{a_s^2(\tau)}{\left( 1 - \frac{n \nu_s(\tau)}{c} \right)^4} + \frac{2 (n \cdot a_s(\tau)) \left( \nu_s(\tau) \cdot a_s(\tau) \right)}{c \left( 1 - \frac{n \nu_s(\tau)}{c} \right)^4} - \frac{\left( 1 - \frac{\nu_s^2(\tau)}{c^2} \right) (n \cdot a_s(\tau))^2}{\left( 1 - \frac{n \nu_s(\tau)}{c} \right)^6} \right],
\]
(5.90)
is the energy radiated in a solid angle in unit time measured with respect to the particle’s motion.

- **points: Problem 5.19:** Perform the angular integration of Eq. (5.90) and show that
\[
\frac{d\mathcal{E}}{d\tau} = \frac{2e^2 a_s^2(\tau)}{3c^3} \left[ \left( 1 - \frac{c^2}{\nu_s^2(\tau)} \right)^{-3} \right].
\]
(5.91)

- **points: Problem 5.20:** Synchrotron radiation: Consider a particle of mass \( m \) and charge \( e \) moving in the presence of a constant magnetic field, \( H \), as in §21 of Landau-Lifshitz, Vol II. You can use the solution for the motion of the particle derived in that section.

- (a): Show that Eq. (5.91) reduces to
\[
\frac{d\mathcal{E}}{d\tau} = \frac{2e^2 H^2 v^2}{3m^2c^5} \left[ 1 - \frac{v^2}{c^2} \right],
\]
(5.92)
where \( v \) is the speed of the particle.

- (b): Assume that
\[
n = \cos \alpha \sin \beta \ \mathbf{x} + \cos \alpha \cos \beta \ \mathbf{y} + \sin \alpha \ \mathbf{z}; \quad \alpha \in [0, \pi]; \quad \beta \in [0, 2\pi],
\]
in Eq. (5.90) to show that the power radiated in a solid angle \( d\Omega \) averaged over one period is
\[
\frac{d\mathcal{E}}{d\tau} = \frac{e^4 H^2 v^2}{8\pi^2 m^2 c^5} \int_0^{2\pi} (\frac{\xi}{c} - \cos \alpha \sin \phi)^2 + \left( 1 - \frac{v^2}{c^2} \right) \sin^2 \alpha \cos^4 \alpha\sin \phi d\phi.
\]
(5.94)

- (c): Perform the integral in Eq. (5.94) and show that
\[
\frac{d\mathcal{E}}{d\Omega} = \frac{e^4 H^2 v^2}{32\pi m^2 c^5} \left[ 8 - 4 \cos^2 \alpha - \frac{v^2}{c^2} \left( 1 + 3\frac{v^2}{c^2} \right) \cos^4 \alpha \right] \frac{1}{\left( 1 - \frac{v^2}{c^2} \cos^2 \alpha \right)^2}.
\]
(5.95)

Plot \( \frac{d\mathcal{E}}{d\Omega} \) as a function of \( \alpha \) for several choices of \( \frac{v^2}{c^2} \). Discuss the behavior from a physical viewpoint.
Chapter 6

An overview of electrodynamics of continuous media

We move from the study of electrodynamics of point like particles to electrodynamics of continuous media. The subject is vast and we will only cover a few basic concepts described in Landau and Lifshitz, Vol 8.

6.1 Electrostatics of conductors

The electric field inside a conductor in the electrostatic case is zero since any electric field inside a conductor will produce a current since the charges are free to move. Since

$$\nabla \cdot \mathbf{E} = 4\pi \rho,$$

it follows that there can be no charge density inside a conductor in the electrostatic case. Any net charge on a conductor call only reside on its surface. Quoting L&L, Thus the problem of electrostatics of conductors amounts to determining the electric field in the vacuum outside the conductors and the distribution of charges on their surfaces.

The relevant Maxwell’s equations in the vacuum for the electric field are

$$\nabla \times \mathbf{E} = 0; \quad \nabla \cdot \mathbf{E} = 0.$$  

We can solve these equations by introducing the electrostatic potential,

$$\mathbf{E} = -\nabla \phi; \quad \Delta \phi = 0; \quad \Delta = \nabla \cdot \nabla.$$  

**Statement:** Electric field at the surface of the conductor has to be normal to the surface.

**Proof:** Consider a point on the surface of the conductor. We can erect local coordinates such that the normal at that point is the $z$ direction and $x$ and $y$ lie along the surface. The electric field normal to the surface around that point, namely, $E_z$, will be a smooth function of point on the surface, namely, $x$ and $y$. That is to say, $\frac{\partial E_z}{\partial x}$ and $\frac{\partial E_z}{\partial y}$ will both be finite in the neighborhood of the point. Since $\nabla \times \mathbf{E} = 0$ everywhere including the surface of the conductor,

$$\left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) x + \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial x} \right) y + \left( \frac{\partial E_x}{\partial x} - \frac{\partial E_x}{\partial y} \right) z = 0.$$  

Therefore, it follows that $\frac{\partial E_z}{\partial x}$ and $\frac{\partial E_x}{\partial z}$ are both finite as we move across the surface. Since there is no electric field inside the conductor ($z < 0$), we conclude that $E_x$ and $E_y$ have to be zero on the surface. Since the electric field tangential to the surface is zero everywhere on the surface, we can conclude that the potential, $\phi$, on the surface of a conductor has to be a constant.

Let $E_n$ denote the electric field at the surface of the conductor where $n$ denotes that it is normal to the conductor. Since $E_n$ is zero inside the conductor, this quantity is discontinuous as we move from the inside of the conductor to the outside of the conductor. From $\nabla \cdot \mathbf{E} = 4\pi \rho$, we know that this discontinuity should result in a surface charge density on the conductor with

$$E_n = 4\pi \sigma,$$  

(6.5)
where $\sigma$ is the surface charge density on the conductor. Integrating over the whole surface of the conductor, we see that the total charge, $e$, on the conductor is given by

$$e = \frac{1}{4\pi} \int E_n dA = -\frac{1}{4\pi} \int \frac{\partial \phi}{\partial n} dA.$$  \hspace{1cm} (6.6)

**Statement:** The electric potential due to an arbitrary number of conductors with arbitrary charges on the surface and with no other charges anywhere else in space cannot reach a maximum of minimum anywhere outside of the conductors.

**Proof:** We will prove this by contradiction. Let us assume that the potential reaches a maximum somewhere outside of the conductors. Let us consider an infinitesimal sphere around this point. We know that $\frac{\partial \phi}{\partial n} < 0$ everywhere on the surface where $n$ denotes the derivative normal to the surface since the potential is a maximum at the center of the sphere. Therefore,

$$0 > \int \frac{\partial \phi}{\partial n} dA = -\int E_n dA.$$ \hspace{1cm} (6.7)

This says that there is a net charge enclosed by the infinitesimal sphere which is a contradiction. By a similar argument, we can also show that the potential cannot reach a minimum anywhere outside of the conductors. Therefore, a test charge placed anywhere outside of all conductors will not be stable.

The total energy stored by placing a set of charges $q_a$, $a = 1, \cdots, n$ on $n$ conductors is given by

$$U = \frac{1}{8\pi} \int E^2 dV$$ \hspace{1cm} (6.8)

where the integration is only over the space outside the conductors since there is no electric field inside the conductors. We can rewrite the expression for the total energy as

$$U = -\frac{1}{8\pi} \int \mathbf{E} \cdot \nabla \phi dV = -\frac{1}{8\pi} \int [\partial_i (E_i \phi) - (\partial_i E_i) \phi] dV$$ \hspace{1cm} (6.9)

Since our integration region is outside the conductors, $\partial_i E_i = \nabla \cdot \mathbf{E} = 0$, everywhere and the second integrand vanishes everywhere. The first integral reduces to a boundary integral over the boundaries of all conductors and the boundary at infinity. Since the electric field vanishes at infinity, we only have to integrate over all the conductor boundaries. Therefore,

$$U = \sum_{a=1}^{N} \frac{1}{8\pi} \int E_n \phi dA_a.$$ \hspace{1cm} (6.10)

The normal $n$ points out of the conductor and nullifies the negative sign. Now we note that the potential at the boundary of a conductor is constant on the surface and we will refer to this constant as $\phi_a$ for the conductor $a$. Using Eq. (6.6), we arrive at

$$U = \frac{1}{2} \sum_{a=1}^{N} \phi_a e_a; \quad e_a = \frac{1}{4\pi} \int E_n dA_a.$$ \hspace{1cm} (6.11)

Let us consider the physical situation where we have forced the potentials on the conductors to be $\phi_a$ by connecting each one to a battery with respect to the ground (point at infinity). We can solve the problem, $\Delta \phi = 0$ with these boundary conditions and find the potential everywhere. From this result, we can compute the net charge $e_a$ induced on all the conductors along with the surface charge distribution. Let $C_{ab}$ be the induced charge on all the conductors when we set $\phi_b = 1$ and all other conductors to be at zero potential. Due to the linearity of the problem we can conclude that

$$e_a = \sum_{b=1}^{N} C_{ab} \phi_b.$$ \hspace{1cm} (6.12)

We will refer to the quantities, $C_{ab}$, as capacitances; either self ($C_{aa}$) or induced ($C_{ab}$, $a \neq b$). Clearly $C$ is a square matrix and assuming it has an inverse,

$$\phi_a = \sum_{b=1}^{N} C_{ab}^{-1} e_b.$$ \hspace{1cm} (6.13)
6.1. ELECTROSTATICS OF CONDUCTORS

Inserting Eq. (6.12) in Eq. (6.11) results in

\[ U = \frac{1}{2} \sum_{a,b=1}^{N} C_{ab} \phi_a \phi_b, \quad (6.14) \]

where as inserting Eq. (6.13) in Eq. (6.11) results in

\[ U = \frac{1}{2} \sum_{a,b=1}^{N} C_{-ab}^{-1} e_a e_b. \quad (6.15) \]

These two results imply that the matrix \( C \) is symmetric and therefore also \( C^{-1} \). Furthermore, we have

\[ \delta U = \sum_{a,b=1}^{N} C_{ab} \delta \phi_a \phi_b = \sum_{a=1}^{N} \delta \phi_a e_a, \quad (6.16) \]

and

\[ \delta U = \sum_{a,b=1}^{N} C_{-ab}^{-1} \delta e_a e_b = \sum_{a=1}^{N} \delta e_a \phi_a. \quad (6.17) \]

**Statement:** \( C_{aa} > 0 \) and \( C_{aa}^{-1} > 0 \) for all \( a \).

**Proof:** We know from the original expression for the total energy, namely Eq. (6.8), that \( U > 0 \). Consider the physical situation where \( \phi_a = 1 \) and \( \phi_b = 0 \) for all \( b \neq a \). That is to say we have applied a unit potential to the \( a \) conductor and grounded all other conductors. The total energy from Eq. (6.14) is

\[ U = \frac{1}{2} C_{aa} > 0 \Rightarrow C_{aa} > 0. \quad (6.18) \]

Consider the physical situation where \( e_a = 1 \) and \( e_b = 0 \) for all \( b \neq a \). This corresponds to the physical situation where all conductors are floating in space and we have induced a net charge only on one conductor, \( a \). The total energy from Eq. (6.15) is

\[ U = \frac{1}{2} C_{aa}^{-1} > 0 \Rightarrow C_{aa}^{-1} > 0. \quad (6.19) \]

**Statement:** \( C_{ab} < 0 \) for all \( a \neq b \).

**Proof:** Consider the physical situation where \( \phi_b = 1 \) and \( \phi_a = 0 \) for all \( a \neq b \). Then, from Eq. (6.12), we have

\[ e_a = C_{ab}. \quad (6.20) \]

We know that \( e_b > 0 \) since \( C_{bb} > 0 \). Furthermore, we know that the potential everywhere outside the conductors is positive. For, if it was negative somewhere it should have a minimum somewhere outside the conductors since \( \phi_b = 1 \) and \( \phi_a = 0 \) for all \( a \neq b \). Therefore, \( \frac{\partial \phi}{\partial n} > 0 \) everywhere on the surface of the conductor, \( a \neq b \). The net surface charge on the conductor, \( a \neq b \) is therefore negative from Eq. (6.6). If \( e_a < 0 \) for \( a \neq b \), it follows that \( C_{ab} < 0 \) for all \( a \neq b \).

- **points: Problem 6.1:** Prove that \( 0 > C_{ab} = -\sqrt{C_{aa} C_{bb}}. \)

- **points: Problem 6.2:** Compute the energy stored in a system of two conductors with capacitances, \( C_{11}, C_{12} \) and \( C_{22} \) for the following cases:

  - (a): Conductor 1 is at a potential \( V \) and conductor 2 is at zero potential.
  - (b): Conductor 1 is at a zero potential and conductor 2 is at a potential \( -V \).
  - (c): Conductor 1 is at a potential \( \frac{V}{2} \) and conductor 2 is at a potential \( -\frac{V}{2} \).
  - (d): Conductor 1 has a net charge of \( e \) and conductor 2 has a net zero charge.
  - (e): Conductor 1 has a net zero charge and conductor 2 has a net charge of \( -e \).
  - (f): Conductor 1 has a net charge of \( \frac{e}{2} \) and conductor 2 has a net charge of \( -\frac{e}{2} \).

Obtain an expression for the mutual capacitance, \( C \), defined either as \( U = \frac{1}{2} CV^2 \) or \( U = \frac{e^2}{2C} \) for all six cases. Discuss your results in the context of what you learned about capacitance in your basic undergraduate physics.
• – points: Problem 6.3: Let \( e_a \) and \( \phi_a \) describe the charges and potential on a set of \( a = 1, \cdots, N \) conductors. Let \( e'_a \) and \( \phi'_a \) describe another instance of the charges and potential on the same set of conductors. Prove that
\[
\sum_{a=1}^{N} \phi'_a e_a = \sum_{b=1}^{N} \phi_b e'_b.
\] (6.21)

• – points: Problem 6.4: A point charge \( e \) is situated at \( O \), near a system of earthed conductors, and induced on them charges \( e_a \). If the charge \( e \) were absent, and the \( a^{th} \) conductor were at a potential \( \phi'_a \), the remainder being earthed, the field at \( O \) would be \( \phi'_0 \). Express the charges \( e_a \) in terms of \( \phi'_a \) and \( \phi'_0 \).

6.2 Electrostatics of dielectrics

Quoting L&L, The fundamental property of dielectrics is that a steady current cannot flow in them. Therefore, there can be electric field and charge density inside a dielectric. The relevant Maxwell’s equations inside the dielectric for the electrostatic case are
\[
\nabla \times E = 0; \quad \nabla \cdot E = 4\pi \rho.
\] (6.22)

Let us begin our discussion by assuming that the net charge on the dielectric is zero,
\[
\int \rho dV = 0. \tag{6.23}
\]
Noting that \( \rho \) outside the dielectric is zero, we will now show that we can consistently find a vector \( \mathbf{P} \) such that
\[
\rho = -\nabla \cdot \mathbf{P} \tag{6.24}
\]
with \( \mathbf{P} = 0 \) everywhere outside the dielectric. Inserting Eq. (6.24) into Eq. (6.23), we obtain
\[
\int \nabla \cdot \mathbf{P} dV = \int \mathbf{P} \cdot d\mathbf{A} = 0. \tag{6.25}
\]
Since the volume integral encloses the whole dielectric, the surface integral uses \( \mathbf{P} \) outside the dielectric which is zero. Therefore, this equation is trivially satisfied. We refer to \( \mathbf{P} \) as the dielectric polarization.

Consider the total dipole moment of the dielectric which need not be zero even though the net charge is zero. The total dipole moment is
\[
d_i = \int x_i \rho dV = -\int x_i (\partial_j P_j) dV = -\int \partial_j (x_i P_j) dV + \int P_j \partial_i x_j dV = \int P_i dV \Rightarrow \mathbf{d} = \int \mathbf{P} dV. \tag{6.26}
\]
Therefore, the polarization vector is the density of the dipole moment inside the dielectric. Inserting Eq. (6.24) in Eq. (6.22) gives us the equation
\[
\nabla \cdot \mathbf{D} = 0; \quad \mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}. \tag{6.27}
\]

6.2.1 Boundary conditions between dielectrics

Consider a boundary between two dielectrics under electrostatic conditions. We need to satisfy \( \nabla \times \mathbf{E} = 0 \) and \( \nabla \cdot \mathbf{D} = 0 \) on either side of the boundary. Assuming that boundary is locally in the \( x-y \) plane we look at
\[
\left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) x + \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) y + \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) z = 0. \tag{6.28}
\]
on the boundary. Since \( E_z \) will be continuous as a function of the point \( (x, y) \) on the boundary, it follows from the \( x \) and \( y \) components of the equation that \( E_x \) and \( E_y \) should be continuous across the boundary. Therefore, tangential components of \( \mathbf{E} \) are continuous across the dielectric boundary. Since
\[
\partial_x D_x + \partial_y D_y + \partial_z D_z = 0 \tag{6.29}
\]
on the boundary and since \( D_x \) and \( D_y \) are continuous as a function of the point \( (x, y) \) on the boundary, it follows that the normal component of \( \mathbf{D} \) is continuous across the dielectric boundary.
6.3. STATIC MAGNETIC FIELD

6.2.2 Dielectric constant

The polarization of a dielectric material is its intrinsic property and it is a response to the presence of an electric field. A simple characterization is one of linear behavior:

\[ P = \kappa E, \tag{6.30} \]

where \( \kappa \) is the polarization coefficient which in general could be a tensor that depends on the location inside the dielectric. If the material is homogeneous and isotropic, \( \kappa \) will reduce a single constant. Therefore,

\[ D = (1 + 4\pi \kappa)E = \varepsilon E; \quad \kappa = \frac{\varepsilon - 1}{4\pi}. \tag{6.31} \]

We refer to \( \varepsilon \) as the dielectric constant in a homogeneous material.

- **points: Problem 6.5:** Show that

\[ \rho = -\frac{1}{4\pi\varepsilon}E \cdot \nabla \varepsilon, \tag{6.32} \]

in a inhomogeneous but isotropic material.

6.3 Static magnetic field

In analogy with a dielectric material, we assume that we have magnetic materials were we can have a locally time independent current density with no net current flowing in or out of the material. Then the relevant equations for the magnetic field are

\[ \nabla \cdot B = 0; \quad \nabla \times B = \frac{4\pi}{c}j; \quad \int j \cdot dA = 0. \tag{6.33} \]

Noting that \( j \) is zero outside the material, we define a vector \( M \) such that

\[ j = \varepsilon \nabla \times M, \tag{6.34} \]

inside the material and \( M \) being zero outside the material. Since the surface integral in \( \int j \cdot dA = 0 \) includes all the material, we see that

\[ \int (\nabla \times M) \cdot dA = \int M \cdot dl = 0, \tag{6.35} \]

is trivially satisfied since the line integral lies just outside the material. Inserting Eq. (6.34) into Eq. (6.33) results in

\[ \nabla \times H = 0; \quad B = H + 4\pi M, \tag{6.36} \]

where we have reintroduced the notation, \( H \), since there is no current density term on the right. Following a logic similar to that for a dielectric boundary, we can conclude that tangential components of \( H \) are continuous across the boundary of a magnetic material and normal component of \( B \) is continuous across the boundary of a magnetic material.

We will now show that \( M \) can be thought of as a magnetic moment density. To see this we compute the total magnetic moment,

\[ m_i = \frac{1}{2c} \int \epsilon_{ijk} x_j \delta_k dV = \frac{1}{2} \int \epsilon_{ijk} x_j \delta_k \partial_l M_l dV = \frac{1}{2} \int [\delta_l \delta_{jn} - \delta_{ln} \delta_{jl}] x_j \partial_l M_n dV = \frac{1}{2} \int [ -M_i + 3M_j ] dV = \int M_i dV. \tag{6.37} \]

Note that the boundary term is zero upon integration by parts since \( M \) is zero outside the material.

Assuming linearity, we write

\[ M = \chi H, \tag{6.38} \]

and refer to \( \chi \) as the magnetic susceptibility. Inserting this into Eq. (6.36) leads to

\[ B = \mu H; \quad \mu = 1 + 4\pi \chi; \quad \chi = \frac{\mu - 1}{4\pi}. \tag{6.39} \]

Note that in general, \( \mu \) can be a tensor that depends on the location inside the material.

\(^1\text{We will change notation and use } B \text{ instead of } H \text{ to emphasize that we are not in vacuum but inside a magnetic material.}\)
6.4 Conductors with a steady current

Consider a conductor in which there is a steady (time independent) current, \( j \). Since there is no time dependence in any quantity, it follows that any charge density has to be independent of time and the continuity equation reduces to

\[
\nabla \cdot j = 0. \tag{6.40}
\]

Furthermore, we also have

\[
\nabla \times E = 0. \tag{6.41}
\]

The presence of a current is a property of the material and assuming linearity, we write

\[
\[ i \] = \sigma_{ik} E_k, \tag{6.42}
\]

where we have allowed for anisotropy and \( \sigma_{ik} \) is the conductivity tensor. Following a logic similar to that for a dielectric boundary, we can conclude that \textbf{tangential components of \( E \) are continuous across the conductor boundary} and \textbf{normal component of \( j \) is continuous across the conductor boundary}.

6.4.1 Hall effect

Assuming an isotropic material and let us apply a constant electric field in the \( x \) direction equal to \( E_x \). Then we have a current in the \( x \) direction given by

\[
\[ j \] = \sigma E_x. \tag{6.43}
\]

Now we apply a constant magnetic field \( H_z \) in the \( z \) direction. This causes a current to flow also in the \( y \) direction given by

\[
\[ j \] = \sigma E_x - \sigma E H_y. \tag{6.44}
\]

The second term is the Hall current and is perpendicular to the electric field that caused the original charge motion. We have therefore introduced an anisotropy by applying a magnetic field and we have \( \sigma_{21}(H) < 0 \). If we repeat the above argument by changing the electric field to \( E_y \), we arrive at

\[
\[ j \] = \sigma E_y + \sigma E H_x. \tag{6.45}
\]

and we conclude that

\[
\sigma_{12}(H) = -\sigma_{21}(H). \tag{6.46}
\]

If we has reversed the sign of the magnetic field instead of the electric field, we would arrive at the conclusion that

\[
\sigma_{21}(-H) = -\sigma_{21}(H); \quad \sigma_{12}(-H) = -\sigma_{12}(H). \tag{6.47}
\]

Therefore, we arrive at the general result that

\[
\sigma_{jk}(H) = \sigma_{kj}(-H). \tag{6.48}
\]