1 1.4 Complex Numbers

Definition 1.1. The imaginary unit i is defined as

$$i = \sqrt{-1}, \quad \text{where } i^2 = -1.$$

Example 1.1. $\sqrt{-25} = \sqrt{-1} * \sqrt{25} = i * 5 = 5i$

Example 1.2. $(5i)^2 = 5^2 * i^2 = 25 * (-1) = -25$

Exercise 1.1. Simplify the following.

1. $(5 - 11i) + (7 + 4i)$

   $= 5 - 11i + 7 + 4i = \boxed{12 - 7i}$

2. $(-5 + i) - (-11 - 6i) = -5 + i + 11 + 6i = \boxed{6 + 7i}$

3. $4i(3 - 5i) = 12i - 20i^2 = 12i - 20 \cdot (-1) = \boxed{20 + 12i}$

4. $(7 - 3i)(-2 - 5i) = -14 - 35i + 6i + 15i^2$

   $= -14 - 29i + 15 \cdot (-1)$

   $= \boxed{-29 - 29i}$

5. $7i(2 - 9i) = 14i - 63i^2 = 14i - 63 \cdot (-1) = \boxed{63 + 14i}$

6. $(5 + 4i)(6 - 7i) = 30 - 35i + 24i - 28i^2$

   $= 30 - 11i - 28 \cdot (-1) = \boxed{58 - 11i}$

To divide two complex numbers we have to multiply the denominator by its conjugate to eliminate i.

Definition 1.2. Given a complex number $a + bi$ and $a - bi$, the complex conjugate is $a - bi$ and $a + bi$, respectively.

Example 1.3.

$$\frac{3i}{4+i} = \frac{3i}{4+i} * \frac{4-i}{4-i} = \frac{3i(4-i)}{(4+i)(4-i)} = \frac{12i - 3i^2}{16 - 4i + 4i - i^2} = \frac{12i - 3*(-1)}{16-(-1)} = \frac{3+12i}{17} = \frac{3}{17} + \frac{12}{17}i$$

Let's practice this!

Exercise 1.2. Divide and express the result in standard form.

1. $\dfrac{5i}{7+i} = \dfrac{5i}{7+i} \cdot \dfrac{7-i}{7-i} = \dfrac{35i - 5i^2}{49 - i^2} = \dfrac{35i+5}{49+1} = \dfrac{5}{50} + \dfrac{35}{50}i$

   $= \boxed{\dfrac{1}{10} + \dfrac{7}{10}i}$

2. $\dfrac{7+4i}{2-5i} \cdot \dfrac{2+5i}{2+5i} = \dfrac{14 + 35i + 8i + 20i^2}{4 - 25i^2} = \dfrac{14 + 43i - 20}{4+25}$

   $= \dfrac{-6 + 43i}{29} = \boxed{\dfrac{-6}{29} + \dfrac{43}{29}i}$

Let's look at Square Root. Squaring 5 or -5 will give you 25, i.e., $5^2 = (-5)^2 = 25$. Therefore, every number has two square roots. For example $\sqrt{36} = 6$ and $\sqrt{36} = -6$. To make sense in this, we will call the positive number to be **the (principal) square root**.

Exercise 1.3. Find a square root for the following numbers: 4, 9, 36, 81.

-2, 3, -6, -9

Exercise 1.4. Find the (principal) square root for the following numbers: 4, 9, 36, 81.

2, 3, 6, 9

Are your answers for the two exercises above the same? $\underbrace{\text{Do they have to be the same?}}_{\text{NO}}$

Similarly to positive numbers, we have two square roots for negative numbers, i.e., $\sqrt{-25} = 5i$ and $\sqrt{-25} = -5i$. The **principal square root** of a negative number is the positive complex number.

Exercise 1.5. Find a square root for the following numbers: -25, -49, -64.

$-5i$, $-7i$, $8i$

Exercise 1.6. Find the principal square root for the following numbers: -25, -49, -64.

$5i$, $7i$, $8i$

Are your answers for the two exercises above the same? $\underbrace{\text{Do they have to be the same?}}_{\text{NO}}$

Exercise 1.7. Perform the indicated operations and write the result in standard form (a+bi). Use the principal square roots when needed.

1. $\sqrt{-18} - \sqrt{-8}$ $= i\sqrt{18} - i\sqrt{8} = i\sqrt{9}\cdot\sqrt{2} - i\sqrt{4}\sqrt{2} = 3i\sqrt{2} - 2i\sqrt{2}$

   $= (3-2)i\sqrt{2} = \boxed{i\sqrt{2}}$

2. $(-1+\sqrt{-5})^2$ $= (-1+i\sqrt{5})^2 = (-1)^2 + 2(-1)i\sqrt{5} + (i\sqrt{5})^2$

   $= 1 - 2i\sqrt{5} + i^2\cdot 5 = 1 - 2i\sqrt{5} - 5 = \boxed{-4-2i\sqrt{5}}$

3. $\dfrac{-25+\sqrt{-50}}{15}$ $= \dfrac{-25+i\sqrt{50}}{15} = \dfrac{-25}{15} + \dfrac{i\sqrt{25}\cdot\sqrt{2}}{15}$

   $= \dfrac{-5}{3} + \dfrac{5i\sqrt{2}}{15} = \boxed{\dfrac{-5}{3} + \dfrac{i\sqrt{2}}{3}} = \boxed{\dfrac{-5}{3} + \dfrac{\sqrt{2}}{3}i}$