1 1.4 Complex Numbers

Definition 1.1. The imaginary unit i is defined as

$$i = \sqrt{-1}, \quad \text{where } i^2 = -1.$$

Example 1.1. $\sqrt{-25} = \sqrt{-1} * \sqrt{25} = i * 5 = 5i$

Example 1.2. $(5i)^2 = 5^2 * i^2 = 25 * (-1) = -25$

Exercise 1.1. Simplify the following.

1. $(5 - 11i) + (7 + 4i)$

2. $(-5 + i) - (-11 - 6i)$

3. $4i(3 - 5i)$

4. $(7 - 3i)(-2 - 5i)$

5. $7i(2 - 9i)$

6. $(5 + 4i)(6 - 7i)$

To divide two complex numbers we have to multiply the denominator by its conjugate to eliminate i.

Definition 1.2. Given a complex number $a + bi$ and $a - bi$, the complex conjugate is $a - bi$ and $a + bi$, respectively.

Example 1.3.

$$\frac{3i}{4+i} = \frac{3i}{4+i} * \frac{4-i}{4-i} = \frac{3i(4-i)}{(4+i)(4-i)} = \frac{12i - 3i^2}{16 - 4i + 4i - i^2} = \frac{12i - 3 * (-1)}{16 - (-1)} = \frac{3 + 12i}{17} = \frac{3}{17} + \frac{12}{17}i$$

Let's practice this!

Exercise 1.2. Divide and express the result in standard form.

1. $\frac{5i}{7+i}$

2. $\frac{7+4i}{2-5i}$

Let's look at Square Root. Squaring 5 or -5 will give you 25, i.e., $5^2 = (-5)^2 = 25$. Therefore, every number has two square roots. For example $\sqrt{36} = 6$ and $\sqrt{36} = -6$. To make sense in this, we will call the positive number to be **the (principal) square root**.

Exercise 1.3. Find a square root for the following numbers: 4, 9, 36, 81.

Exercise 1.4. Find the (principal) square root for the following numbers: 4, 9, 36, 81.

Are your answers for the two exercises above the same? Do they have to be the same?

Similarly to positive numbers, we have two square roots for negative numbers, i.e., $\sqrt{-25} = 5i$ and $\sqrt{-25} = -5i$. The **principal square root** of a negative number is the positive complex number.

Exercise 1.5. Find a square root for the following numbers: -25, -49, -64.

Exercise 1.6. Find the principal square root for the following numbers: -25, -49, -64.

Are your answers for the two exercises above the same? Do they have to be the same?

Exercise 1.7. Perform the indicated operations and write the result in standard form (a+bi). Use the principal square roots when needed.

1. $\sqrt{-18} - \sqrt{-8}$

2. $(-1 + \sqrt{-5})^2$

3. $\dfrac{-25 + \sqrt{-50}}{15}$

2 1.5 Quadratic Equations

Definition 2.1. The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is

$$b^2 - 4ac.$$

The discriminant will tell you how many solutions does a quadratic equation have. If the discriminant is **positive**, the equation have **two real** solutions. If the discriminant is **zero**, the equation have **one real** solution and if the discriminant is negative, the equation has **two complex** solutions.

Example 2.1. Use the discriminant to find the number and type of solutions:

1. $x^2 + 4x - 5 = 0$
   The discriminant is $4^2 - 4 * 1 * (-5) = 16 + 20 = 36$. Since it is positive, the equation will have two real solutions.

2. $x^2 - 2x + 2 = 0$
   The discriminant is $(-2)^2 - 4 * 1 * 2 = 4 - 8 = -4$. Since it is negative, the equation will have two complex solutions.

3. $x^2 + 2x + 1 = 0$
   The discriminant is $2^2 - 4 * 1 * 2 = 4 - 4 = 0$. Since it is zero, the equation will have one real solution.

Exercise 2.1. Use the discriminant to find the number and type of solutions:

1. $3x^2 + 4x - 5 = 0$

2. $9x^2 - 6x + 1 = 0$

3. $3x^2 - 8x + 7 = 0$

3 1.6 Other types of equations

3.1 Solving a Polynomial Equation by Factoring

Exercise 3.1. Solve by factoring:

1. $3x^4 = 27x^2$

2. $x^3 + x^2 = 4x + 4$

3.2 Radical Equations

To solve an equation with radicals follow these steps:

1. Arrange terms, so that one radical is isolated on one side of the equation.
2. Eliminate the radical on one side using exponentiation on both side of the equation.
3. If the equation still contains radical, go to step 1.
4. Solve the equation and check all proposed solutions in the original equation.

Example 3.1. Solve $\sqrt{x+3}+3=x$.

$$\begin{aligned}
\sqrt{x+3}+3 &= x \\
\sqrt{x+3} &= x-3 \\
(\sqrt{x+3})^2 &= (x-3)^2 \\
x+3 &= x^2-6x+9 \\
x+3-x^2+6x-9 &= 0 \\
-x^2+7x-6 &= 0 \\
x^2-7x+6 &= 0 \\
(x-6)(x-1) &= 0
\end{aligned}$$

$$\begin{aligned}
x-6=0 &\qquad x-1=0 \\
x=6 &\qquad x=1
\end{aligned}$$

We found that the proposed solutions are 6 and 1. But we have to plug them in to the original equation:

$$\begin{aligned}
x=6: \quad & \sqrt{6+3}+3=6 \\
& \sqrt{9}+3=6 \\
& 3+3=6 \\
& true
\end{aligned}$$

$$\begin{aligned}
x=1: \quad & \sqrt{1+3}+3=1 \\
& \sqrt{4}+3=1 \\
& 2+3=1 \\
& false
\end{aligned}$$

Therefore the only solution for this equation is $x=6$.

Exercise 3.2. Solve $\sqrt{2x-1}+2=x$.

Exercise 3.3. Solve $\sqrt{2x-1}+2=x$.

3.3 Equations That Are Quadratic in Form

Some equations that are not quadratic can be written as quadratic equations using an appropriate substitution.

Given Equation	Substitution	New Equation
$x^4 - 10x^2 + 9 = 0$ $(x^2)^2 - 10(x^2) + 9 = 0$	$u = x^2$	$u^2 - 10u + 9 = 0$
$5x^{2/3} + 11x^{1/3} + 2 = 0$ $5\left(x^{1/3}\right)^2 + 11(x^{1/3}) + 2 = 0$	$u = x^{1/3}$	$5u^2 + 11u + 2 = 0$

Example 3.2. Solve: $x^4 - 10x^2 + 9 = 0$.

We can use the substitution $x^2 = u$ since

$$x^4 - 10x^2 + 9 = (x^2)^2 - 10(x^2) + 9$$

to get the equation

$$\begin{aligned} u^2 - 10u + 9 &= 0 \\ (u-9)(u-1) &= 0 \end{aligned}$$

Now we have $u = 9$ and $u = 1$. But we have to find the solution for x! To do it, we look at the substitution:

$$\begin{array}{cc} x^2 = u = 9 & x^2 = u = 1 \\ x^2 = 9 & x^2 = 1 \\ x = \pm\sqrt{9} & x = \pm\sqrt{1} \\ x = \pm 3 & x = \pm 1 \end{array}$$

The solution set is $\{3, -3, 1, -1\}$.

Exercise 3.4. Solve $x^4 - 8x^2 - 9 = 0$

Exercise 3.5. Solve $5x^{\frac{2}{3}} + 11x^{\frac{1}{3}} + 2 = 0$

Exercise 3.6. Solve $(x^2 - 5)^2 + 3(x^2 - 5) - 10 = 0$