Section 2.7

Ex: $f(x) = 2x$, $g(x) = \frac{1}{2}x$

$$f(g(x)) = f \circ g(x) = f(\tfrac{1}{2}x) = 2 \cdot (\tfrac{1}{2}x) = \cancel{2} \cdot \tfrac{1}{\cancel{2}} x = \underline{\underline{x}}$$

$$g(f(x)) = g(2x) = \tfrac{1}{2}(2x) = \underline{\underline{x}}$$

f and g are inverses of each other, i.e., g is the inverse function of f and f is the inverse function of g.

Ex: Are $f(x) = 2x - 3$, $g(x) = \frac{1}{2}x + 3$ inverses?

$f(g(x)) \stackrel{?}{=} x$ $\longleftrightarrow$ $f(g(t)) = t$

$$f(g(x)) = f(\tfrac{1}{2}x + 3) = 2 \cdot (\tfrac{1}{2}x + 3) - 3 = 2 \cdot \tfrac{1}{2}x + 2 \cdot 3 - 3$$
$$= \underbrace{x + 3 \neq x}_{g \text{ is not the inverse of } f}$$

Definition

Let f and g be two functions such that

$f(g(x)) = x$, for every x in the domain of g

and

and

$g(f(x)) = x$, for every x in the domain of f.

The function g is the inverse of f and is denoted as f^{-1}. Thus $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Ex: let $f(x) = 3x+2$, is $f^{-1}(x) = \frac{x-2}{3}$?

$$f(f^{-1}(x)) = f\left(\frac{x-2}{3}\right) = \cancel{3}\cdot\left(\frac{x-2}{\cancel{3}}\right)+2 = x-2+2 = \underline{\underline{x}} \checkmark$$

$$f^{-1}(f(x)) = f^{-1}(3x+2) = \frac{(3x+2)-2}{3} = \frac{\cancel{3}x}{\cancel{3}} = \underline{\underline{x}} \checkmark$$

$\boxed{\text{Yes}}$

Ex: find $f^{-1}(x)$ if $\begin{cases} f(x) = 3x+5. \\ f^{-1}(x) = \frac{x-5}{3} \end{cases}$ ✓

doesn't have an inverse

Dom

$(-\infty, \infty)$ $\quad f(x) = 2x^2+5$

$[5, \infty)$ $\quad f^{-1}(x) = \sqrt{\frac{x-5}{2}}$

not inverses

$$f^{-1}(f(x)) = f^{-1}(2x^2+5) = \sqrt{\frac{(2x^2+5)-5}{2}} = \sqrt{\frac{8x^2}{8}}$$

$$= \sqrt{x^2} = |x| \neq x$$

let's try $x = -1$

$$f^{-1}(f(-1)) = f^{-1}(2(-1)^2+5) = f^{-1}(7) = \sqrt{\frac{7-5}{2}} = \sqrt{1} = \underline{1}$$

How do find the inverse of $f(x)$:

1) Replace $f(x)$ with y
2) Interchange x and y
3) Solve the eq. for y.
   If the equation doesn't define y as a function of x, $f(x)$ doesn't have an inverse.
4) Replace y with $f^{-1}(x)$.

__Ex:__ Find $f^{-1}(x)$, $f(x) = 3x+5$

$$y = 3x+5 \rightarrow x = 3y+5$$

$$x-5 = 3y$$

$$\boxed{\frac{x-5}{3}} = y$$

$$\boxed{f^{-1}(x) = \frac{x-5}{3}}$$

__Ex:__ find $f^{-1}(x)$, $f(x) = 2x^2+5$, domain of $f = (-\infty, \infty)$

$$y = 2x^2+5 \longleftrightarrow x = 2y^2+5$$

$$x-5 = 2y^2$$

not a function

$$x-5=2y^2$$

$$\sqrt{\frac{x-5}{2}}=\sqrt{y^2}$$

$$\boxed{\sqrt{\frac{x-5}{2}}=|y|}$$

not a function

f doesn't have an inverse.

<u>Ex:</u> $f(x)=x^3+1$, find $f^{-1}(x)$.

$$y=x^3+1 \Longleftrightarrow x=y^3+1$$

$$x-1=y^3$$

$$\sqrt[3]{x-1}=\sqrt[3]{y^3}=y$$

$$\boxed{f^{-1}(x)=\sqrt[3]{x-1}}$$

<u>Ex:</u> $f(x)=x+x^2$, $f^{-1}(x)=?$

$$y=x+x^2 \Longleftrightarrow x=y+y^2=y(1+y)$$

not solvable $\Rightarrow$ f^{-1} doesn't exist

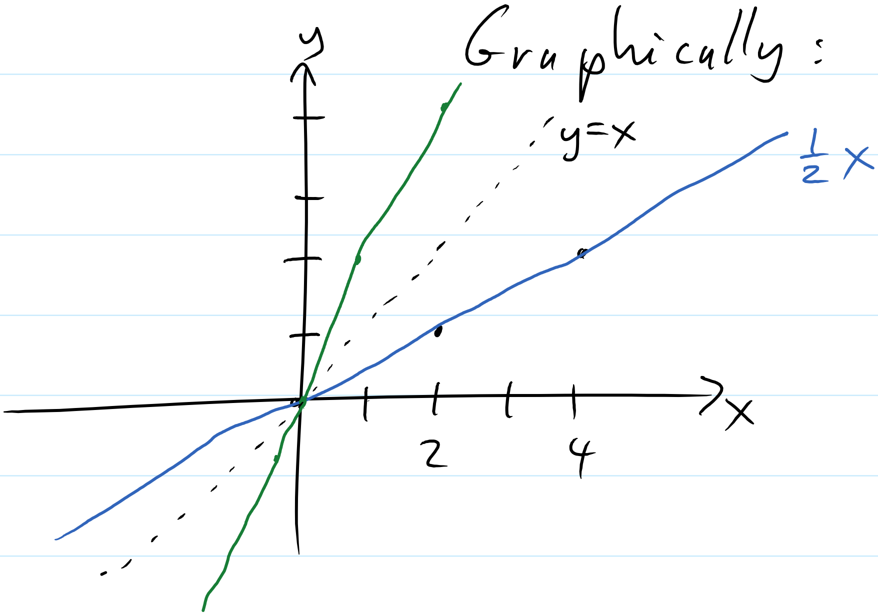
<u>Ex:</u> $f(x)=x+x^3$, $f^{-1}(x)=?$

$$y=x+x^3 \Longleftrightarrow x=y+y^3=y(1+y^2)$$

not solvable. :(

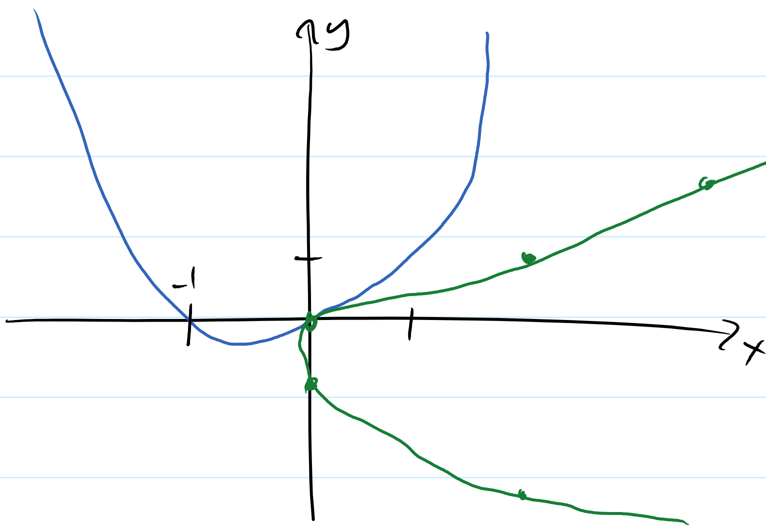
but f^{-1} exists

Graphically:



f (x,y)	f^{-1} (x,y)
$(2,1)$	$(1,2)$
$(4,2)$	$(2,4)$
$(0,0)$	$(0,0)$
$(-1,-\frac{1}{2})$	$(-\frac{1}{2},-1)$

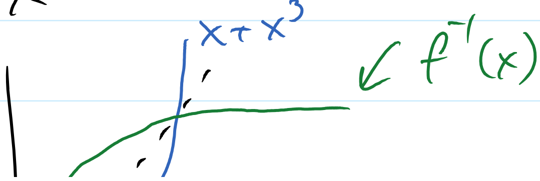
$f(x) = x + x^2$

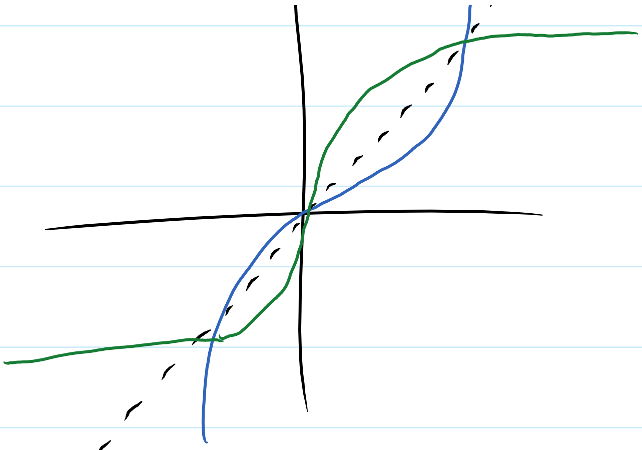


f (x,y)	f^{-1} (x,y)
$(-2,2)$	$(2,-2)$
$(-1,0)$	$(0,-1)$
$(0,0)$	$(0,0)$
$(1,2)$	$(2,1)$
$(2,6)$	$(6,2)$

not a function b/c vertical line test.

$f(x) = x + x^3$

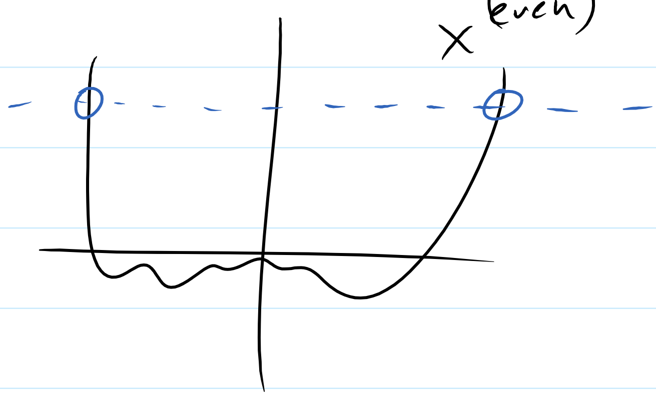




Horizontal line test

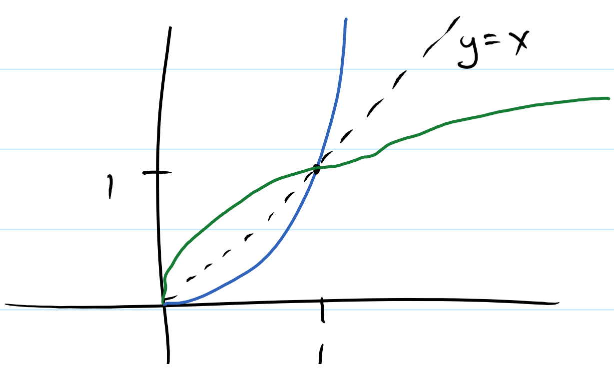
A function f has an inverse, f^{-1}, if there is no horizontal line that intersect the graph of the function f at more than one point.

$x^{(even)}$

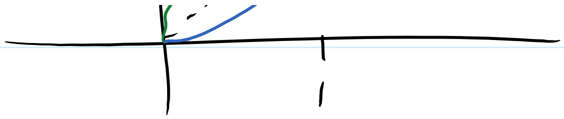


$x^{(even)}$ doesn't have an inverse.

$f(x) = x^2$ domain $[0, \infty)$



$\sqrt{x} = f^{-1}(x)$ domain $[0, \infty)$



Note:

	f	f^{-1}
domain	$\square$	$\square$
range	$\square$	$\square$