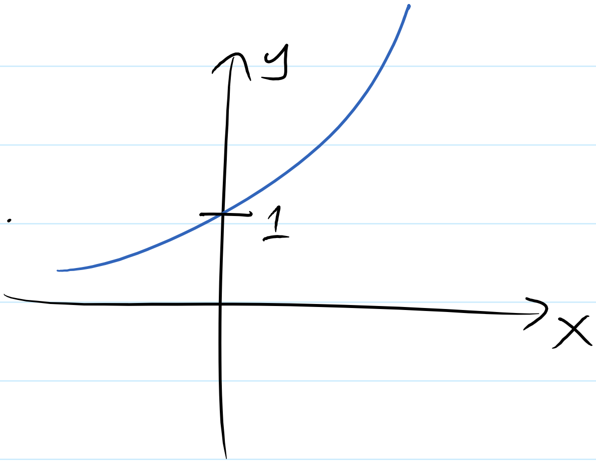
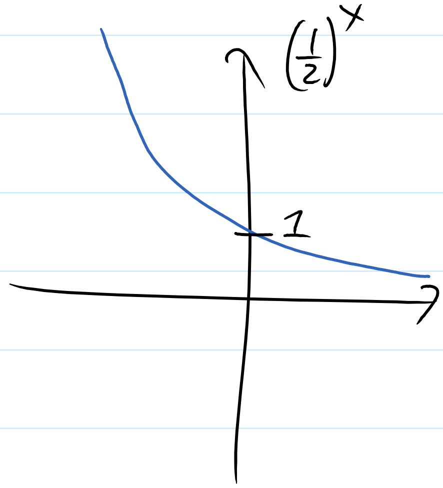
Section 4.1

$f(x)=e^x$

$e \approx 2.718...$





$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Future Value of an Investment

Suppose a principal P is invested at annual interest rate r, for t years to accumulat to a future value $B(t)$.

If interest is compounded k times per year, then

$$B(t) = P\left(1+\frac{r}{k}\right)^{kt},$$

if the interest is compounded continuously, then

$$B(t) = Pe^{rt}$$

$$B(t) = Pe^{\cdot\cdot}$$

Ex: Suppose \$1000 is invested at an annual interest rate of 6% for 10 years. Find the future value if the interest is compounded:

a) Quaterly: $k=4$

$$B(10) = 1000\left(1+\frac{0.06}{4}\right)^{4\cdot 10} = 1000\left(1+\frac{0.06}{4}\right)^{40}$$

$$= \$1814.02$$

b) monthly: $k=12$

$$B(10) = 1000\left(1+\frac{0.06}{12}\right)^{12\cdot 10} = \$1819.4$$

c) continuously:

$$B(10) = 1000e^{0.06\cdot 10} = 1000e^{0.6} = \$1822.12$$

Effective Interest Rate Formulas:

If interest is compounded at the nominal rate r, the effective interest rate is the simple annual interest rate r_e that yields the same interest rate after 1 year.

The formula:

$$r_e = (1+i)^k - 1 \quad \ldots \quad i = \frac{r}{k}$$

The formula:

$$r_e = (1+i)^k - 1, \quad \text{where } i = \frac{r}{k}$$
$$= \left(1+\frac{r}{k}\right)^k - 1$$

while continuous compounding yields

$$r_e = e^r - 1$$

Ex: Which is better, an investment that earns 10% compounded quarterly, one that 9.95% compounded monthly, or one that earns 9.9% compounded continuously.

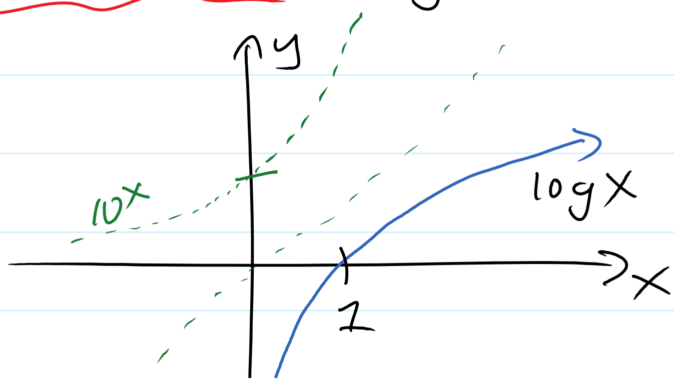
Let's compare r_e for each scenario:

1) $r_e = \left(1+\frac{0.1}{4}\right)^4 - 1 = 0.1038$

2) $r_e = \left(1+\frac{0.0995}{12}\right)^{12} - 1 = 0.10416$

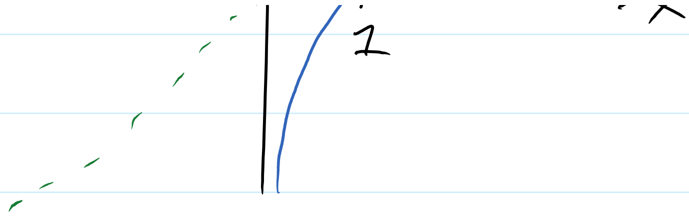
3) $r_e = e^{0.099} - 1 = 0.104066$

Section 4.2 (log. functions)



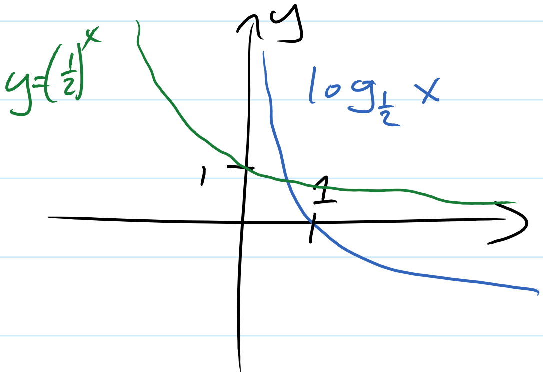
$\log x = \log_{10} x$

base



<u>Def:</u> $\log_b x = c \iff b^c = x$

<u>Rule:</u> $\log_b (x)^a = a \cdot \log_b x$



<u>Rule:</u> $\log_b b^a = a \cdot \log_b b = a$

<u>Def:</u> "Natural logarithm"

$$\log_e x = \ln x$$

$$\ln e^x = x.$$

<u>Rule:</u> $\log_b a = \dfrac{\ln a}{\ln b} = \dfrac{\log_c a}{\log_c b}$, for any positive c. $(c \neq 1)$

<u>Ex:</u> How long will it take $5000 to grow to

Ex: How long will it take \$5000 to grow to \$7000 in an investment at annual rate 6% if the compounding is:

a) Quarterly: $(k=4)$

$$B(t) = 5000\left(1+\frac{0.06}{4}\right)^{4t}$$

$$7000 = 5000\left(1+\frac{0.06}{4}\right)^{4t}$$

$$1.4 = 1.015^{4t} \quad / \ln$$

$$\ln(1.4) = \ln(1.015^{4t})$$

$$\ln(1.4) = 4t \cdot \ln(1.015)$$

$$t = \boxed{\frac{\ln(1.4)}{4\ln(1.015)}} \approx \boxed{5.6498}$$

b) continuously

$$B(t) = 5000\, e^{0.06t}$$

$$7000 = 5000\, e^{0.06t}$$

$$1.4 = e^{0.06t} \quad / \ln$$

$$\ln 1.4 = (0.06t)\ln e$$

$$\ln 1.4 = 0.06t$$

$$t = \boxed{\frac{\ln 1.4}{0.06}} \approx 5.6078$$

Doubling time: A quantity $Q(t) = Q_0 e^{kt}$, $(k>0)$ doubles when

$$t = \frac{\ln 2}{k}$$

Half-time: A quantity $Q(t) = Q_0 e^{-kt}$, $(k>0)$ halves when

$$t = \frac{\ln 2}{k}$$

Ex: How long will it take for a quantity A_0 to triple in value if it is invested at an annual interest rate r, comp. continuously.

$$B(t) = Pe^{rt}$$

$$3A_0 = A_0 e^{rt}$$

$$3 = e^{rt} \qquad / \ln ..$$

$$\ln 3 = \ln(e^{rt})$$

$$\ln 3 = rt \cdot \underbrace{\ln e}_{=1}$$

$$t = \boxed{\frac{\ln 3}{}}$$

$$t = \boxed{\frac{\ln 3}{r}}$$