Section 7.3

The Second Partials Test

Let $f(x,y)$ be a function whose partial derivatives, $f_x, f_y, f_{xx}, f_{yy}, f_{xy}$ all exist, and let $D(x,y)$ be the function

$$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

1) Find all criticals points of $f(x,y)$, i.e., all points (a,b) when $f_x(a,b) = 0 = f_y(a,b)$

2) For each critical point (a,b), evaluate $D(a,b)$.

3) If $D(a,b) < 0$, the point (a,b) is a saddle pt.

4) If $D(a,b) > 0$, find the value of $f_{xx}(a,b)$:

   a) $f_{xx}(a,b) > 0$, the pt (a,b) is a rel. minimum.

   b) $f_{xx}(a,b) < 0$, the pt (a,b) is a rel. maximum

5) If $D(a,b) = 0$, the test is inconclusive, the pt (a,b) can be rel. min/max or a saddle pt.

Ex: Determine if crit. pts are min/max/saddle pt.

$$f(x,y) = x^2 + y^2$$

$f_x = 2x$ $\qquad$ $f_{xx} = 2$

$f_y = 2y$ $\qquad$ $f_{yy} = 2$

$f_{xy} = 0$

Crit. pts $\begin{cases} 2x = 0 \to x = 0 \\ 2y = 0 \to y = 0 \end{cases}$ $(0,0)$

$D(x,y) = 2 \cdot 2 - 0^2 = 4$

$D(0,0) = 4 > 0 \to$ either min or max

$f_{xx}(0,0) = 2 > 0 \to$ $\boxed{(0,0) \text{ is a rel. min}}$

- $f(x,y) = 12x - x^3 - 4y^2$

$f_x = 12 - 3x^2$ $\qquad$ $f_{xx} = -6x$

$f_y = -8y$ $\qquad$ $f_{yy} = -8$

$f_{xy} = 0$

Crit. pts: $\begin{cases} 12 - 3x^2 = 0 \to x^2 = \frac{12}{3} = 4 \to x = \pm 2 \\ -8y = 0 \to y = 0 \end{cases}$

Crit. pts: $\begin{cases} \text{[illegible]} \\ -8y = 0 \rightarrow \underline{y=0} \end{cases}$

$$\boxed{(-2,0),\ (2,0)}$$

$$D(x,y) = -6x\cdot(-8) - 0^2 = 48x$$

$\boxed{(-2,0)}$ $D(-2,0) = 48\cdot(-2) = \underline{-96 < 0}$

$\boxed{(-2,0) \text{ is a saddle pt.}}$

$\boxed{(2,0)}$ $D(2,0) = 48\cdot 2 > 0$ ✓

$f_{xx}(2,0) = -6\cdot 2 = \underline{-12 < 0}$

$\boxed{(2,0) \text{ is a rel. max}}$

- $f(x,y) = x^3 - y^3 + 6xy$

$f_x = 3x^2 + 6y$ $\qquad f_{xx} = 6x$

$f_y = -3y^2 + 6x$ $\qquad f_{yy} = -6y$

$\qquad f_{xy} = 6$

Crit. pts: $\begin{cases} 3x^2+6y=0 \\ -3y^2+6x=0 \end{cases}$ $\longrightarrow$ $-3y^2=-6x$

$x=\frac{1}{2}y^2$

$$3\left(\tfrac{1}{2}y^2\right)^2+6y=0$$

$$3\cdot\tfrac{1}{4}\cdot y^4+6y=0$$

$$3y\left(\tfrac{1}{4}y^3+2\right)=0$$

$$\tfrac{3}{4}y\left(y^3+8\right)=0$$

$\underline{y=0}$ $\swarrow$ $x=\frac{1}{2}(0)^2=0$ $\boxed{(0,0)}$

$y^3+8=0$

$y^3=-8$

$\underline{y=-2}$

$\to x=\frac{1}{2}(-2)^2=\frac{1}{2}\cdot 4=\underline{2}$

$\boxed{(2,-2)}$

$$D=6x(-6y)-6^2=-36xy-36$$

$\boxed{(0,0):}$

$D(0,0)=-36\cdot 0\cdot 0-36=\underline{-36<0}\to$ $\boxed{\text{(0,0) is a saddle pt}}$

$\boxed{(2,-2)}$

$D(2,-2)=-36\cdot 2\cdot(-2)-36=\underline{36\cdot 4-36>0}$

$f_{xx}(2,-2) = 6\cdot 2 = \underline{12} > 0$

$\rightarrow$ $\boxed{(2,-2) \text{ is a rel. min}}$