MAC2311, Summer 2015

Final Exam

July 29, 2015

Name Key

- You will be told when to begin the work and when to terminate work on the examination. You must stop when instructed. Points may be deducted in case of violations.
- Please show your work to support your answers that require calculations. Correct but unsupported answers may not be given full credit.
- The use of a cell phone or other electronic communication devices during the examination is not allowed. The exam will be canceled and a grade of "0" will be assigned to anyone who opens a cell phone during the examination or if one is found on their seat or hand.

No graphing calculators are allowed

1. (10 points)

a) Find the average rate of change of $\tan(x)$ over the interval $[0, \frac{\pi}{3}]$

$$\frac{\Delta y}{\Delta x} = \frac{\tan\frac{\pi}{3} - \tan 0}{\frac{\pi}{3} - 0} = \frac{\sqrt{3} - 0}{\pi/3} = \boxed{\frac{3\sqrt{3}}{\pi}}$$

b) Find the equation for a tangent line to $\tan(x)$ at $x = \frac{\pi}{3}$

$[\tan(x)]' = \sec^2 x$

slope: $\sec^2\frac{\pi}{3} = \frac{1}{\cos^2\frac{\pi}{3}} = \frac{1}{(\frac{1}{2})^2} = 4$

$$y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right)$$

$$\boxed{y = 4x - \frac{4\pi}{3} + \sqrt{3}}$$

2. Find the limits

a) (5 points) $\lim_{x\to 0^-} \frac{e^x}{x\sqrt{x^2+4}}$ $= \lim_{x\to 0^-} \frac{1}{x\sqrt{4}} = \boxed{-\infty}$

b) (5 points) $\lim_{x\to 2} \frac{-1}{x-2}$

$$\left.\begin{aligned} \lim_{x\to 2^-} \frac{-1}{x-2} &= \infty \\ \lim_{x\to 2^+} \frac{-1}{x-2} &= -\infty \end{aligned}\right\} \lim_{x\to 2} \frac{-1}{x-2} \ \boxed{\text{DNE}}$$

3. (10 points) Find the point (x, y), at which the graph of $y = \frac{\ln x}{x^2}$ has a horizontal tangent.

$$y' = \frac{\frac{1}{x}x^2 - 2x \cdot \ln x}{x^4} = \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3} = 0$$

$$1 - 2\ln x = 0$$

$$\ln x = \frac{1}{2}$$

$$x = e^{1/2} = \sqrt{e}$$

$$y(\sqrt{e}) = \frac{\ln e^{1/2}}{(\sqrt{e})^2} = \frac{\frac{1}{2}\ln e}{e} = \frac{1}{2e}$$

$$(x, y) = \boxed{\left(\sqrt{e}, \frac{1}{2e}\right)}$$

4. (10 points) Find the first and second derivatives.

a) $y = e^{x^3}$

$$y' = e^{x^3} \cdot 3x^2 = \boxed{3x^2 e^{x^3}}$$

$$y'' = 3\left[2x\, e^{x^3} + x^2 e^{x^3} 3x^2\right] = 6x e^{x^3} + 3 \cdot 3 x^2 x^2 e^{x^3}$$

$$= \boxed{6x e^{x^3} + 9x^4 e^{x^3}}$$

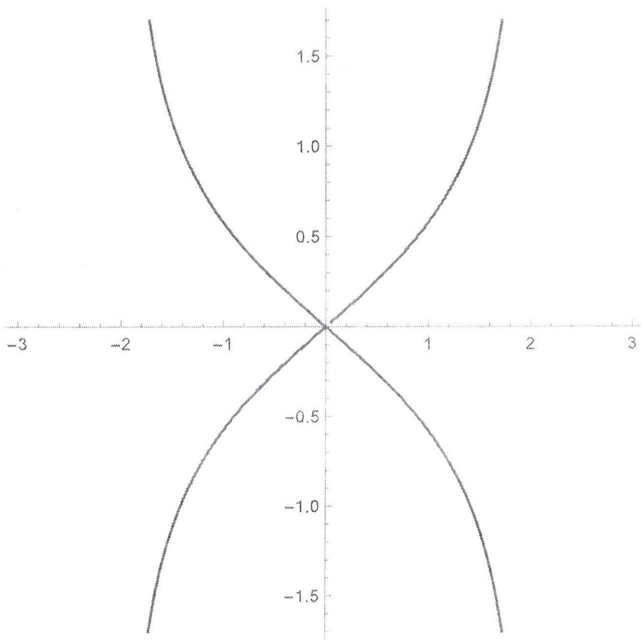
b) $y = \sec x$

$$y' = \boxed{\sec x \cdot \tan x}$$

$$y'' = \sec x \tan x \tan x + \sec^2 x \sec x$$

$$= \boxed{\sec x \tan^2 x + \sec^3 x}$$

5. (10 points) Graph of $x^2y^2 + x^2 = 4y^2$ is depicted below. Use implicit differentiation to find $y' = \frac{dy}{dx}$.



$$x^2y^2 + x^2 = 4y^2$$

$$2xy^2 + 2yy'x^2 + 2x = 4 \cdot 2yy'$$

$$y'(2yx^2 - 8y) = -\cancel{2}xy^2 - \cancel{2}x$$

$$y' = \boxed{\frac{-xy^2 - x}{yx^2 - 4y}}$$

or

$$y' = \frac{-2xy^2 - 2x}{2yx^2 - 8y} = \frac{2xy^2 + 2x}{8y - 2yx^2}$$

6. (10 points) Use logarithmic differentiation to find the derivative of y as a function of x.

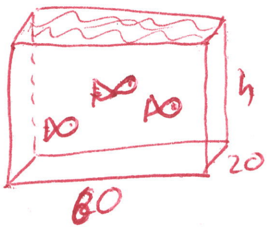
$$y = x^x$$

$$\ln y = \ln x^x = x \cdot \ln x$$

$$\frac{1}{y} y' = \ln x + \frac{1}{x} \cdot x = \ln x + 1$$

$$y' = y(\ln x + 1) = \boxed{x^x(\ln x + 1)}$$

7. (10 points) A rectangular fish tank is being filled at the constant rate of 40cm^3/sec. The base of the tank has dimensions 20×60cm. What is the rate of change of the height of water in the tank?



$$V' = \frac{dV}{dt} = 40$$

$$V = 60 \cdot 20 \cdot h$$

$$V = 1200h$$

$$\frac{dV}{dt} = 1200 \frac{dh}{dt}$$

$$h' = \frac{dh}{dt} = \frac{1}{1200} \cdot \frac{dV}{dt} = \frac{1}{1200} \cdot 40 = \boxed{\frac{1}{30} \text{ cm/sec}}$$

8. (10 points) Find the intervals on which the function is increasing or decreasing.

$$y = \frac{2x}{x^2 + 4}$$

$$y' = \frac{2(x^2+4) - 2x \cdot 2x}{(x^2+4)^2} = \frac{2(x^2+4-2x^2)}{(x^2+4)^2} = \frac{2(4-x^2)}{(x^2+4)^2} = 0$$

$$4 - x^2 = 0$$

$$x = \pm 2$$

	$(-\infty,-2)$	$(-2,2)$	$(2,\infty)$
$f'(x)$	$-$	$+$	$-$
	↘	↗	↘

$f(x)$ is increasing on $(-2,2)$
$f(x)$ is decreasing on $(-\infty,-2)$ and $(2,\infty)$

9. (10 points) Determine all critical points for the function. Determine which is local minimum or maximum.

$$y = 3x^2 - 96\sqrt{x}.$$

$y' = 6x - \dfrac{96}{2\sqrt{x}} = 6x - 48x^{-1/2} = 6x^{-1/2}(x^{3/2} - 8) = 0$

Domain: $[0, \infty)$

$x^{3/2} - 8 = 0$

$x^{3/2} = 8$

$x = 4$

y' DNE if $x = 0$

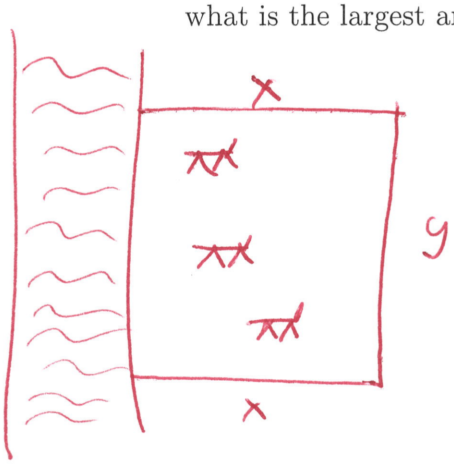
	$(0,4)$	$(4,\infty)$
$f'(x)$	$-$	$+$

$\boxed{\text{C.P: } x=0,\ x=4}$

$x = 0$ is local max

$x = 4$ is local min.

10. (10 points) A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 2000m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?



$2000 = 2x + y \rightarrow y = 2000 - 2x$

$A = x \cdot y$

$A = x(2000 - 2x)$

$= 2000x - 2x^2$

$A' = 2000 - 4x = 0$

$4x = 2000$

$x = 500$

$y = 2000 - 2 \cdot 500$

$y = 1000$

$A(500, 1000) = \boxed{500{,}000\ m^2}$

$\boxed{500 \times 1000\ m}$

11. (10 points) Evaluate the integral

$$\int_{\sqrt{2}}^{1} \frac{u^7}{2} - \frac{1}{u^3}\, du = \int_{\sqrt{2}}^{1} \frac{1}{2}u^7 - u^{-3}\, du$$

$$\int_{\sqrt{2}}^{1} \frac{1}{2}u^7 - u^{-3}\, du = \frac{1}{2}\frac{1}{8}u^8 - \left(\frac{1}{-2}\right)u^{-2}\Big|_{\sqrt{2}}^{1} = \frac{1}{16}u^8 + \frac{1}{2}u^{-2}\Big|_{\sqrt{2}}^{1} = \frac{1}{16}1^8 + \frac{1}{2}\frac{1}{1^2} - \left(\frac{1}{16}(\sqrt{2})^8 + \frac{1}{2}\frac{1}{2}\right)$$

$$\frac{1}{16} + \frac{1}{2} - \left(\frac{1}{16}\frac{16}{1} + \frac{1}{4}\right) = \frac{1}{16} + \frac{1}{2} - 1 - \frac{1}{4} = \frac{1+8-16-4}{16} = \boxed{\frac{-11}{16}}$$

12. (10 points) Find the derivative

$$\frac{d}{dx}\int_{e^x}^{6} \ln(x)\, dx = \frac{-d}{dx}\int_{6}^{e^x} \ln(x)\, dx$$

$$-\ln(e^x)\cdot e^x = -x\cdot \ln e \cdot e^x = \boxed{-x\cdot e^x}$$

13. (10 points) Evaluate the integral

$$\int \frac{6x^2}{\sqrt{1+2x^3}}\, dx$$

$u = 1+2x^3$

$du = 6x^2\, dx$

$$\int \frac{6x^2}{\sqrt{1+2x^3}}\, dx = \int \frac{1}{\sqrt{u}}\, du = \int u^{-1/2}\, du = 2\cdot u^{1/2} + C$$

$$= \boxed{2\cdot\sqrt{1+2x^3} + C}$$

14. (10 points) Evaluate the integral

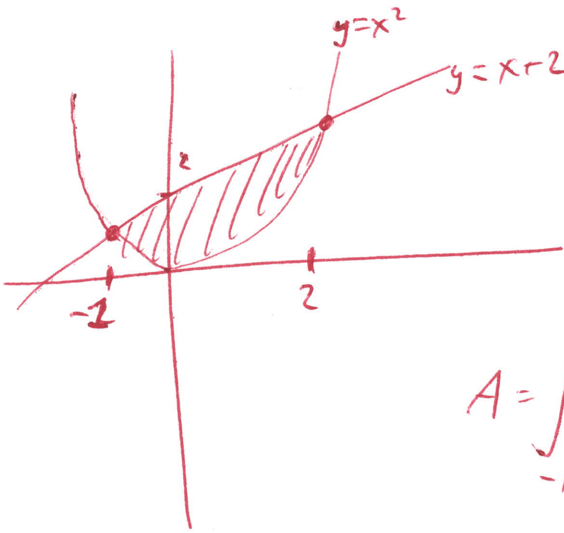
$$\int \sin^4 x \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$\int \sin^4 x \cos x \, dx = \int u^4 \, du = \frac{1}{5} u^5 + C = \boxed{\frac{1}{5} \cdot \sin^5 x + C}$

15. (10 points) Find the area enclosed by $y = x + 2$ and $y = x^2$.



$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$A = \int_{-1}^{2} x + 2 - x^2 \, dx = \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \Big|_{-1}^{2}$

$= \frac{1}{2} \cdot 4 + 2 \cdot 2 - \frac{1}{3} 2^3 - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$

$= 2 + 4 - \frac{8}{3} - \left(\frac{-7}{6}\right) = \frac{10}{3} + \frac{7}{6} = \boxed{\frac{9}{2} = 4.5}$