Section 3.1

Ex: $f(x) = \dfrac{x^2}{x-2}$, Dom: $(-\infty, 2) \cup (2, \infty)$

Critical points: ($f'(x) = 0$ or $f'(x)$ DNE)

$$f'(x) = \frac{(x-2)\cdot 2x - x^2 \cdot 1}{(x-2)^2} = \frac{2x^2 - 4x - x^2}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2}$$

$$= \frac{x(x-4)}{(x-2)^2}$$

$x(x-4) = 0$

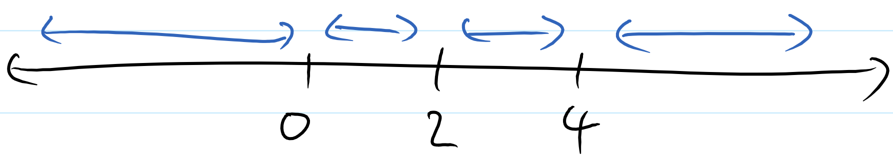
$\underline{x = 0}$, $\underline{x = 4}$

$(x-2)^2 = 0$

$\underline{x = 2}$

(A critical number has to be in the domain)

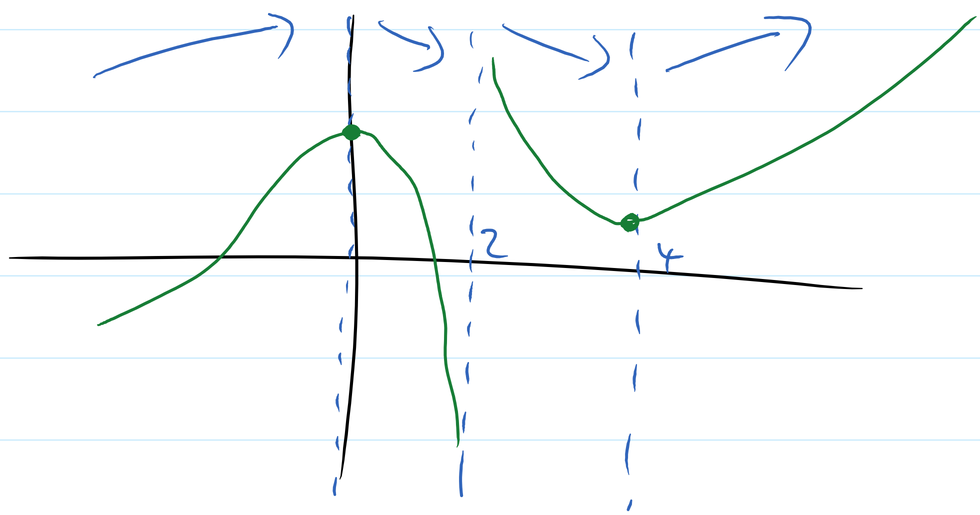
Crit. numbers: 0, 4



f'	$(-\infty, 0)$	$(0, 2)$	$(2, 4)$	$(4, \infty)$
point	-1	1	3	5
$x(x-4)$	$+$	$-$	$-$	$+$
$(x-2)^2$	$+$	$+$	$+$	$+$
Sign of f'	$+$ $\nearrow$	$-$ $\searrow$	$-$ $\searrow$	$+$ $\nearrow$

f is
- increasing on $(-\infty, 0), (4, \infty)$
- decreasing on $(0, 2), (2, 4)$

f might look like:



$x = 0$ - relative maximum
$x = 4$ - relative minimum

Def: The graph of the function $f(x)$ is said to have a relative maximum (minimum) at

$x = c$, if $f(c) \geq f(x)$ for all x in some
$(f(c) \leq f(x))$
interval $a < x < b$ containing c.

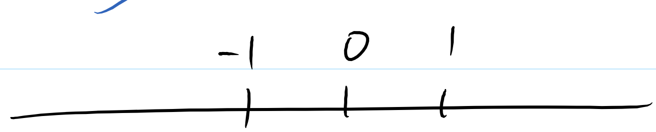
Note: Relative extrema can only occur at critical points.

Thm: (The first derivative test)
A critical point $x = c$ is a:
- relative maximum of $f(x)$ if $f'(x)$ changes from + to − at c.
- relative minimum of $f(x)$ if $f'(x)$ changes from − to + at c.

Ex: Find intervals of inc/dec, relative extrema of: $f(x) = 2x^4 - 4x^2 + 3$

Dom: $(-\infty, \infty)$

$$\left.\begin{aligned} f'(x) = 8x^3 - 8x &= 0 \\ 8x(x^2-1) &= 0 \\ 8x(x-1)(x+1) &= 0 \end{aligned}\right\} \text{Crit. num: } x = 0, 1, -1$$



f'	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
	-2	$-1/2$	$1/2$	2
$8x$	$-$	$-$	$+$	$+$
$(x-1)$	$-$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$	$+$
	$-$ ↘	$+$ ↗	$-$ ↘	$+$ ↗

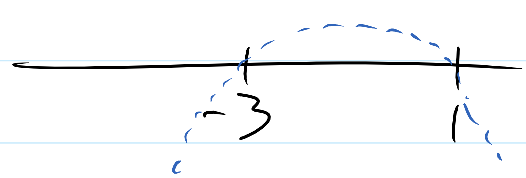
$\left.\begin{matrix} x=-1 \\ x=1 \end{matrix}\right\}$ relative min.

$x=0 \}$ rel. max.

Ex: $g(t)=\sqrt{3-2t-t^2}$

Find intervals of inc/dec + rel. extremas.

$$3-2t-t^2 \geq 0$$
$$-(t^2+2t-3) \geq 0$$
$$-(t-1)(t+3) \geq 0$$



Domain: $[-3,1]$

$$f(t)=(3-2t-t^2)^{1/2}$$

$$f'(t)=\frac{1}{2}(3-2t-t^2)^{-1/2}\cdot(3-2t-t^2)'$$

$$f'(t) = \frac{1}{2}(3-2t-t^2)^{-\frac{1}{2}} \cdot (3-2t-t^2)'$$

$$= \frac{1}{2}\underbrace{(3-2t-t^2)^{-\frac{1}{2}}} \cdot (-2-2t)$$

$3-2t-t^2=0$

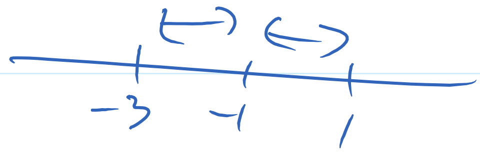
$t=-3, 1$

$-2-2t=0$

$-2t=2$

$t=-1$

Crit. num: $\pm 1, -3$



f'	$(-3,-1)$	$(-1,1)$
$(3-2t-t^2)^{-\frac{1}{2}}$	-2 $+$	0 $+$
$-2-2t$	$+$	$-$
	$+$	$-$

$$\boxed{\begin{array}{l}\text{rel. max: } x=-1 \\ \text{rel. min: none}\end{array}}$$

Ex: A revenue function:

$$R(t) = \frac{63t - t^2}{t^2+63}, \quad 0 \le t \le 63$$

for what t is revenue maximum?

$$R'(t) = \frac{(t^2+63)(63-2t) - (63t-t^2)\cdot 2t}{(t^2+63)^2}$$

$$R'(t) = \frac{\quad}{(t^2+63)^2}$$

$$= \frac{63t^2 - \cancel{2t^3} + 63^2 - 126t - 126t^2 + \cancel{2t^3}}{(t^2+63)^2}$$

$$= \frac{-63t^2 - 126t + 63^2}{(t^2+63)^2} = \frac{-63(t^2+2t-63)}{(t^2+63)^2}$$

$$= \frac{-63(t-7)(t+9)}{(t^2+63)^2}$$

$t = 7, \cancel{-9}$

Dom: $[0, 63]$

$t^2 = -63$

$\times$

R'	$(0,7)$	$(7,63)$
-63	$-$	$-$
$t-7$	$-$	$+$
$t+9$	$+$	$+$
$(t^2+63)^2$	$+$	$+$
	$+$	$-$
	$\nearrow$	$\searrow$

the revenue is maximum at $t=7$.

Section 3.2 (concavity and points of inflection)

Def: If the function $f(x)$ is differentiable on (a,b) then the graph of f is

• concave up if $f''(x) > 0$ on (a,b)

on (a,b) then the graph of f is

- concave up, if $f''(x) > 0$ on (a,b)
- concave down, if $f''(x) < 0$ on (a,b)

Def: An inflection point of the function f is a point $(c, f(c))$ on the graph of f, if the concavity of f changes around c.

Ex: Find inflection pts and concav. intervals

$$f(x) = 3x^5 - 5x^4 - 1$$

$$f'(x) = 15x^4 - 20x^3$$

$$f''(x) = 15 \cdot 4x^3 - 60x^2$$

$$f''(x) = 0$$

$$15 \cdot 4x^3 - 60x^2 = 0$$

$$15x^2(4x - 4) = 0 \qquad x = 0$$

$$60x^2(x - 1) = 0 \qquad x = 1$$

0 1

f''	$(-\infty, 0)$	$(0,1)$	$(1, \infty)$
	-1	$\frac{1}{2}$	2
$60x^2$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$

$x-1$	$-$	$-$	$+$
	$-$	$-$	$+$
	$\cap$	$\cap$	$\cup$

$\boxed{x=1 \text{ is an inflection pt}}$

Ex: Find int of inc/dec, concave up/down, relative extrema, inflection pts:

$$f(x) = \left(\frac{x}{x+1}\right)^2 = \frac{x^2}{(x+1)^2}$$

$$f'(x) = 2\left(\frac{x}{x+1}\right) \cdot \frac{(x+1)\cdot 1 - x\cdot 1}{(x+1)^2} = \frac{2x}{x+1} \cdot \frac{\cancel{x}+1-\cancel{x}}{(x+1)^2}$$

$$= \frac{2x}{(x+1)^3}$$

$2x=0$

$x=0$ ↑ crit. num.

$x+1=0$

$x=-1$

$$f''(x) = \frac{(x+1)^3 \cdot 2 - 2x\cdot 3(x+1)^2 \cdot 1}{\left((x+1)^3\right)^2}$$

$$= \frac{2(x+1)^3 - 6x(x+1)^2}{(x+1)^6} = \frac{\cancel{(x+1)^2}\left[2(x+1) - 6x\right]}{(x+1)^{\cancel{6}\,4}}$$

$$= \frac{-4x+2}{(x+1)^4}$$

$-4x+2=0$

$x=\frac{1}{2}$

$x+1=0$

$x=-1$

f'	$(-\infty,-1)$	$(-1,0)$	$(0,\infty)$

f''	$(-\infty,-1)$	$(-1,\frac{1}{2})$	$(\frac{1}{2},\infty)$