Section 3.4

Def: (Price Elasticity of Demand)
If $q = D(p)$ units of commodity are demanded by the market at the unit price p, where D is a differentiable function, then the price elasticity of demand is given by

$$E(p) = -\frac{p}{q} \cdot \frac{dq}{dp} = -\frac{p \cdot q'}{q}$$

and it has the interpretation:

$E(p) \approx$ [percentage rate of decrease in demand q, produced by a 1% increase in price]

Ex: $q = 240 - 2p$ (for $0 \le p \le 120$)

a) Find $E(p)$
b) Interpret $E(100)$ and $E(50)$

a) $E(p) = -\frac{p \cdot q'}{q} = \frac{-p \cdot (-2)}{240-2p} = \frac{2p}{240-2p} = \boxed{\frac{p}{120-p}}$

b) $E(100) = \frac{100}{120-100} = \frac{100}{20} = \boxed{5}$

If the current price is 100 and we increase it by 1% the demand will decrease by 5%.

$$E(50) = \frac{50}{120-50} = \frac{50}{70} \approx \boxed{0.71}$$

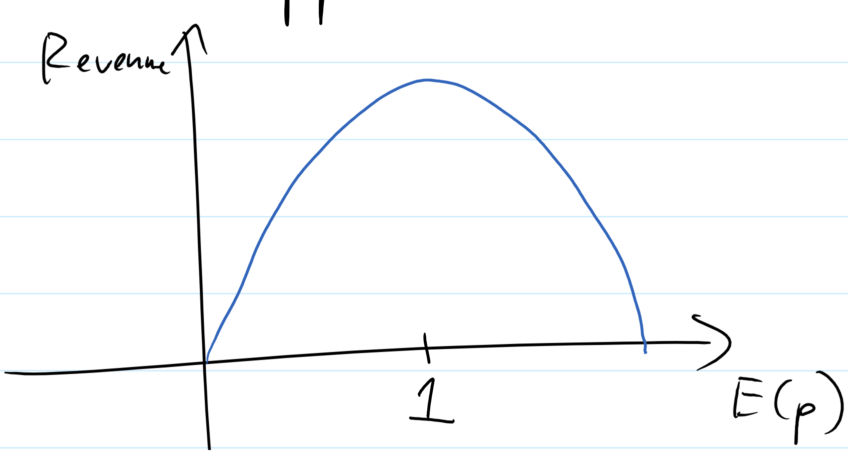
If the price is 50 and we increase it by 1%, the demand will decrease by 0.71%.

levels of elasticity

$E(p) > 1$: **Elastic demand**; The demand is relatively sensitive to changes in price. Revenue ↘ as price ↗

$E(p) < 1$: **Inelastic demand**; The demand is relatively insensitive to changes in price. Revenue ↗ as price ↗

$E(p) = 1$: **Unitary demand**; The percentage changes in price and demand are approx. equal. Revenue → as price ↗.



	1	$E(p)$

Ex: The manager of a store determines that the daily demand for a certain book is $q = 300 - p^2$, where $0 \le p \le \sqrt{300}$

a) Determine where the demand is elastic, inelastic and unitary.

$$E(p) = -\frac{p \cdot q'}{q} = -\frac{p \cdot (300-p^2)'}{300-p^2} = \frac{-p(-2p)}{300-p^2}$$

$$= \frac{2p^2}{300-p^2}$$

Unitary: $\frac{2p^2}{300-p^2} = 1$

$$\frac{2p^2}{300-p^2} - 1 \cdot \frac{300-p^2}{300-p^2} = 0$$

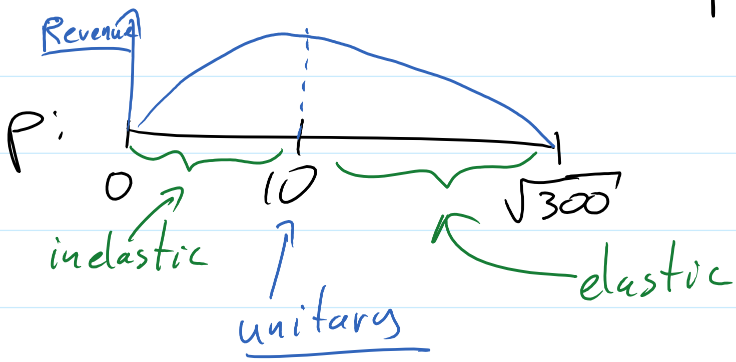
$$\frac{2p^2 - (300-p^2)}{300-p^2} = 0$$

$$\frac{3p^2 - 300}{300-p^2} = 0$$

$$\frac{3(p^2-100)}{300-p^2} = 0 \qquad p^2 - 100 = 0$$

$$p = \pm 10$$

$$\frac{3(p^2-100)}{300-p^2} = 0 \qquad p^2-100=0$$
$$p = \pm 10$$



$$E(1) = \frac{2\cdot 1^2}{300-1^2} = \frac{2}{299} < 1$$

Section 3.5 - Optomization

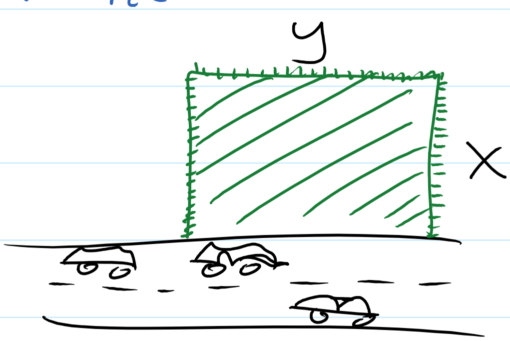
EXAMPLE 3.5.1 Minimizing Amount of Fence

The highway department is planning to build a picnic park for motorists along a major highway. The park is to be rectangular with an area of 5,000 square yards and is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing required for this job? How long and wide should the park be for the fencing to be minimized?

what to minimize

① Understand the problem:

$$x \cdot y = 5000$$



$$f(x) = 2x + y$$

minimize

② Do some algebra to have a function of only one variable

$$x \cdot y = 5000 \rightarrow y = \frac{5000}{x}$$

$$f(x) = 2x + \frac{5000}{x} = 2x + 5000x^{-1}$$

③ Optimize (find min/max) of the function

$$f'(x) = 2 - 5000x^{-2} = 0$$

$$2 - \frac{5000}{x^2} = 0$$

$$\frac{2x^2}{x^2} - \frac{5000}{x^2} = 0$$

$$\frac{2x^2 - 5000}{x^2} = 0$$

$$2x^2 - 5000 = 0$$

$$x^2 = 2500$$

$$x = \pm\sqrt{25}\cdot\sqrt{100}$$

$$= \pm 50$$

$$x^2 = 0$$

x has to be positive

-50 0 50

f'	$(0, 50)$	$(50, \infty)$
x^2	+	+
$2x^2 - 5000$	−	+
f'	−	+

60

$\boxed{x=50}$ rel. min.

$y=\frac{5000}{x}$ $\qquad y=\frac{5000}{50}=\boxed{100}$

Dim: $\boxed{50\times 100}$

EXAMPLE 3.5.2 Maximizing Profit

Mateo owns a small company that makes souvenir T-shirts. He can produce the shirts at a cost of \$2 apiece. The shirts have been selling for \$5 apiece, and at this price, tourists have been buying 4,000 shirts a month. Mateo plans to raise the price of shirts and expects that for each \$1 increase in price, 400 fewer shirts will be sold each month. What price should Mateo charge per shirt to maximize profit?

price ... p. $\qquad$ P= Revenue - Cost

To find the cost, we need the price (2) and quantity.

price	quantity
4	$4000+400=4400$
5	4000
6	$4000-400=3600$
5.5	$4000-400\cdot(5.5-5)$ $=3800$
p	$4000-400\cdot(p-5)$

$$\begin{aligned}C(p)&=2\cdot \text{quantiti}\\&=2(4000-400\cdot(p-5))\\&=8000-800(p-5)\\&=8000-800p+4000\\&=12000-800p\end{aligned}$$

$$\begin{aligned}R(p)&=\text{price}\cdot\text{quantity}\\&=p\cdot(4000-400(p-5))\\&=p(6000-400p)\\&=6000p-400p^2\end{aligned}$$

$$\begin{aligned}\text{Profit}=P(p)&=R(p)-C(p)\\&=6000p-400p^2-(12000-800p)\end{aligned}$$

$$= 6000p - 400p^2 - (12000 - 800p)$$
$$= 6000p - 400p^2 - 12000 + 800p$$
$$= \boxed{-400p^2 + 6800p - 12000}$$

$$P'(p) = -800p + 6800 = 0$$
$$p = \frac{-6800}{-800} = \frac{68}{8} = \frac{34}{4} = \frac{17}{2} = \boxed{8.5}$$

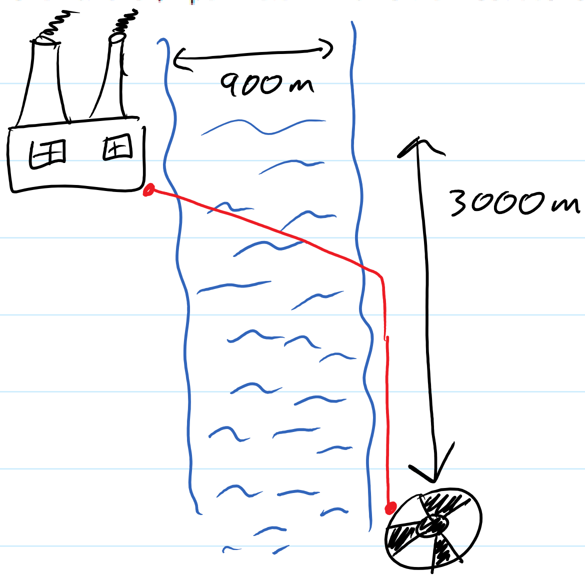
$$P''(p) = -800 < 0$$

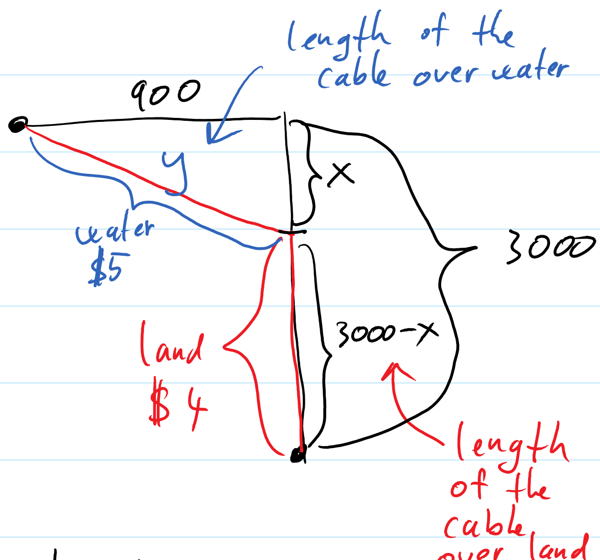
Using the second derivative test, $p = 8.5$ is the max.

$$\boxed{\$8.50}$$

EXAMPLE 3.5.5 Minimizing Cost of Construction

A cable is to be run from a power plant on one side of a river 900 meters wide to a factory on the other side, 3,000 meters downstream. The cost of running the cable under the water is \$5 per meter, while the cost over land is \$4 per meter. What is the most economical route over which to run the cable?





$$\text{Total cost} = \underbrace{\text{land cost}}_{4(3000-x)} + \underbrace{\text{water cost}}_{5 \cdot y}$$

$$\text{Total cost} = \underbrace{\text{land cost}}_{4(3000-x)} + \underbrace{\text{water cost}}_{5\cdot y}$$

We need to minimize $T(x) = 4x + 5y$.
To have a function of one variable, let's use the Pythagorean thm:

$$y^2 = 900^2 + x^2$$

$$y = \sqrt{900^2 + x^2}$$

$$T(x) = 4x + 5(900^2 + x^2)^{1/2}$$

$$T'(x) = 4 + 5 \cdot \frac{1}{\cancel{2}}(900^2 + x^2)^{-1/2} \cdot \cancel{2}x$$

$$= 4 + \frac{5x}{\sqrt{900^2 + x^2}}$$

Let's find minimum:

$$T'(x) = 0$$

$$\frac{4\sqrt{900^2 + x^2} + 5x}{\sqrt{900^2 + x^2}} = 0$$

$$4\sqrt{900^2+x^2}+5x=0$$

$$\left(\sqrt{900^2+x^2}\right)^2=\left(\frac{5}{4}x\right)^2$$

$$900^2+x^2=\frac{25}{16}x^2$$

$$900^2=\frac{9}{16}x^2$$

$$x^2=900^2\cdot\frac{16}{9}$$

$$x^2=1440000$$

$$x=\pm 1200$$

Since x is length, then $x=1200$. The length of the cable over land is $3000-1200=\boxed{1800\text{ m}}$

The length of the cable over water is $\sqrt{900^2+x^2}$
$=\sqrt{900^2+1200^2}$
$=\boxed{1500\text{ m}}$