Section 7.1 - Functions of Several variables.

Def: A function f of the two independent variables x and y is a rule that assigns to each ordered pair (x,y) in a given set D (the domain of f) exactly one real number, denoted by $f(x,y)$.

Ex: $$f(x,y) = \frac{3x^2+5y}{x-y}$$

- Find
  - $f(1,0) = \frac{3\cdot 1+5\cdot 0}{1-0} = \frac{3}{1} = 3$
  - $f(1,-2) = \frac{3\cdot 1+5(-2)}{1-(-2)} = \frac{3-10}{1+2} = \frac{-7}{3}$
  - $f(1,1)$ DNE
- Domain of f: $x-y \neq 0$

  $x \neq y$

  $\{(x,y) \mid x \neq y\}$

let: $f(x,y) = xe^y + \ln x$

• Domain: $x > 0$, $\{(x,y) \mid x > 0\}$

• $f(e^2, \ln 2) = e^2 \cdot e^{\ln 2} + \ln e^2 = e^2 \cdot 2 + 2$
$= 2e^2 + 2$

$f(x,y,z) = xy + xz + yz$

Find $f(-1,2,5) = (-1)\cdot 2 + (-1)\cdot 5 + 2\cdot 5$
$= -2 - 5 + 10 = \boxed{3}$

Ex: A store sells two kind of tenis rockets
The consumer demand for each brand is affected by the pricing of competitors rocket.
The demand for the first kind is

$D_1 = 300 - 20x + 30y$, where x is its price

$D_2 = 200 + 40x - 10y$, where y is its price

Find the store's total annual revenue from the sale of these ractets.

$$R(x,y) = D_1 \cdot x + D_2 \cdot y$$
$$= (300 - 20x + 30y)x + (200 + 40x - 10y)y$$
$$= 300x - 20x^2 + 30xy + 200y + 40xy - 10y^2$$
$$= \boxed{-20x^2 - 10y^2 + 70xy + 300x + 200y}$$

Section 7.2

Note: if $f(x)$ is given, then

$$\frac{d}{dx}f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now: What if $f(x,y)$

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Def: Suppose $z = f(x,y)$. The partial derivative of f with respect to x (y) is denoted by

$$\frac{\partial z}{\partial x} \text{ or } f_x(x,y) \qquad \frac{\partial z}{\partial y} \text{ or } f_y(x,y),$$

and is the function obtained by differentiat. f with respect to x (y), treating y (x) as a constant.

Ex: $f(x,y) = x^2 + 2xy^2 + \frac{2y}{3x}$,

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 2xy^2 + \frac{2y}{3x}\right)$$

$$= \frac{\partial}{\partial x}\left(x^2 + 2y^2 \cdot x + \frac{2y}{3}x^{-1}\right)$$

$$= 2x + 2y^2 \cdot 1 + \frac{2y}{3} \cdot (-1)x^{-2}$$

$$= \boxed{2x + 2y^2 - \frac{2y}{3x^2}}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 2xy^2 + \frac{2y}{3x}\right)$$

$$= \frac{\partial}{\partial y}\left(x^2 + 2x \cdot y^2 + \frac{2}{3x} \cdot y \right)$$

$$= 0 + 2x \cdot 2y + \frac{2}{3x} \cdot 1$$

$$= \boxed{4xy + \frac{2}{3x}}$$

Ex: $z = (x^2 + xy + y)^5$

Find:

$$\frac{\partial z}{\partial x} = \left[(x^2 + xy + y)^5 \right]_x$$

$$= 5(x^2 + xy + y)^4 \cdot \frac{\partial}{\partial x}(x^2 + xy + y)$$

$$= 5(x^2 + xy + y)^4 \cdot (2x + y + 0)$$

$$= \boxed{5(2x+y)(x^2 + xy + y)^4}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left((x^2 + xy + y)^5 \right)$$

$$= 5(x^2 + xy + y)^4 \frac{\partial}{\partial y}(x^2 + xy + y)$$

$$= 5(x^2 + xy + y)^4 \cdot (0 + x \cdot 1 + 1)$$

$$= \boxed{5(x+1)(x^2 + xy + y)^4}$$

$$= \boxed{5(x+1)\,(x^2+xy+y)^4}$$

Ex: $f(x,y) = x e^{-2xy}$

$$f_y = \frac{\partial}{\partial y}\left(x e^{-2xy}\right) = x \cdot \frac{\partial}{\partial y}\left(e^{-2xy}\right)$$

$$= x \cdot e^{-2xy} \cdot \frac{\partial}{\partial y}(-2xy)$$

$$= x e^{-2xy} \cdot (-2x \cdot 1) = \boxed{-2x^2 e^{-2xy}}$$

$$f_x = \frac{\partial}{\partial x}\big(\underbrace{x e^{-2xy}}_{\text{prod. rule}}\big) = 1 e^{-2xy} + x \cdot \frac{\partial}{\partial x}\left(e^{-2xy}\right)$$

$$= e^{-2xy} + x e^{-2xy} \cdot \frac{\partial}{\partial x}(-2y \cdot x)$$

$$= e^{-2xy} + x e^{-2xy} \cdot (-2y \cdot 1)$$

$$= \boxed{e^{-2xy}(1 - 2xy)}$$

Ex: The weekly output of a plant is given by the function

$$Q(x,y) = 1200x + 500y + x^2y - x^3 - y^2 \text{ units,}$$

where x is the number of skilled workers, y is the # of unskilled workers. Currently, y is the # of unskilled workers. Currently, there are 30 skilled workers and 60 unskilled workers. Use marginal analysis to estimate the change in weekly output if 1 additional skilled worker is hired.

Find $Q_x(30, 60)$

$$Q_x(x, y) = 1200 + 0 + y \cdot 2x - 3x^2 - 0$$
$$= 1200 + 2xy - 3x^2$$

$$Q_x(30, 60) = 1200 + 2 \cdot 30 \cdot 60 - 3 \cdot 30^2$$
$$= 1200 + 3600 - 2700$$
$$= \boxed{2100}$$

What if we hire one more unskilled worker?

$Q_y(30, 60)$

$$Q_y = \frac{\partial}{\partial y}\left(1200x + 500y + x^2 y - x^3 - y^2\right)$$

$$= 0 + 500 + x^2 \cdot 1 - 0 - 2y$$
$$= 500 + x^2 - 2y$$

$$Q(30, 60) = 500 + 30^2 - 2 \cdot 60$$
$$= 500 + 900 - 120$$
$$= \boxed{1280}$$