Section 7.2

Def: Two commodities are called substitute if an increase in the demand for either results in a decrease in demand for the other.
Two commodities are called complementary if an increase in the demand for either results in an increase in demand for the other.

substitute: MAC/PC
complementary: burgers/fries,

Let D_1 and D_2 be demands for two commodities, where $D_1(p_1 p_2)$ and $D_2(p_1, p_2)$, p_1 ... price of the first com.
p_2 ... price of the second com.

Naturally,

$$\frac{\partial D_1}{\partial p_1} < 0 \quad \text{and} \quad \frac{\partial D_2}{\partial p_2} < 0.$$

Two commodities are substitute (complimentary) if

$$\frac{\partial D_1}{\partial p_2} > 0 \quad \text{and} \quad \frac{\partial D_2}{\partial p_1} > 0 \qquad \frac{\partial D_1}{\partial p_2} < 0 \quad \text{and} \quad \frac{\partial D_2}{\partial p_1} < 0$$

<u>Ex:</u> Suppose $D_1(p_1, p_2) = 500 + \frac{10}{p_1+2} - 5p_2$

$$D_2(p_1,p_2) = 400 - 2p_1 + \frac{7}{p_2+3}$$

Find if the two commodities are complimentary, substitute or neither.

$$\frac{\partial D_1}{\partial p_2} = 0 + 0 - 5 = -5$$

$$\frac{\partial D_2}{\partial p_1} = 0 - 2 + 0 = -2$$

They are complementary

Second derivatives:

Let $z = f(x,y)$ be a function. We have four second derivatives:

$$1) \ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

2) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

3) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$

4) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$

} The mixed derivative of f.

→ $f_{xy} = f_{yx}$

Find all four second derivatives of

$$f(x,y) = xy^3 + 5xy^2 + 2x + 1$$

$$f_x = \frac{\partial}{\partial x}(y^3 \cdot x + 5y^2 x + 2x + 1) = y^3 \cdot 1 + 5y^2 \cdot 1 + 2$$
$$= y^3 + 5y^2 + 2$$

$$f_y = x\,3y^2 + 5x \cdot 2y + 0 + 0 = 3xy^2 + 10xy$$

$$f_{xx} = \frac{\partial}{\partial x}(y^3 + 5y^2 + 2) = \boxed{0}$$

$$f_{yy} = \frac{\partial}{\partial y}(3xy^2 + 10xy) = 3x\,2y + 10x = \boxed{6xy + 10x}$$

$$f_{xy} = \frac{\partial}{\partial y}(y^3 + 5y^2 + 2) = \boxed{3y^2 + 10y}$$

$$f_{yx} = \frac{\partial}{\partial x}(3xy^2 + 10xy) = \boxed{3y^2 + 10y}$$

Section 7.3 – Relative min/max of $f(x,y)$

Def: The point (x_0, y_0) is a critical point if $\frac{\partial f}{\partial x}(x_0,y_0)=0$ $\frac{\partial f}{\partial y}(x_0,y_0)=0$

The Second Partials Test

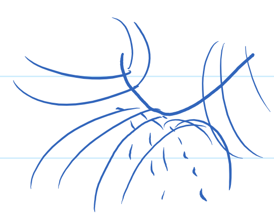
Let $f(x,y)$ be a function whose partial derivatives $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$ all exist, and let $D(x,y)$ be a function

$$D(x,y) = f_{xx}(x,y)\cdot f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$$

1) Find all critical points of $f(x,y)$, i.e. find all (a,b) such that
$f_x(a,b)=0$ $f_y(a,b)=0$

2) For each critical point (a,b) evaluate $D(a,b)$.

3) If $D(a,b)<0$, then (a,b) is a saddle pt.



4) If $D(a,b)>0$, compute $f_{xx}(a,b)$, then

• If $f_{xx}(a,b) > 0$, (a,b) is a relative min.

• if $f_{xx}(a,b) < 0$, (a,b) is a rel. maximum.

If $D(a,b) = 0$, the test is inconclusive and f may have either a rel. extremum or a saddle pt at (a,b).

<u>Ex:</u> $f(x,y) = x^2+y^2$, classify each crit. pt as rel. min/max or a saddle pt.

$f_x = 2x$ $\quad$ $f_{xx} = 2$

$f_y = 2y$ $\quad$ $f_{yy} = 2$

$f_{xy} = 0$

<u>Crit pts:</u> $2x = 0$ $\quad$ $2y = 0$

$x = 0$ $\quad$ $y = 0$

$(0,0)$

$D(x,y) = f_{xx} \cdot f_{yy} - f_{xy}^2$

$= 2 \cdot 2 - 0^2 = 4$

$D(0,0) = 4 > 0 \rightarrow f_{xx}(0,0) = 2 > 0$

(0,0) is a rel. min

Do the same as above for

$f(x,y) = 12x - x^3 - 4y^2$

$f_x = 12 - 3x^2$ $\quad f_{xx} = -6x$

$f_y = -8y$ $\quad f_{yy} = -8$

$f_{xy} = 0$

Crit. pts:

$12 - 3x^2 = 0$ $\quad -8y = 0$

$12 = 3x^2$ $\quad y = 0$

$4 = x^2$

$x = \pm 2$

(2,0) , (-2,0)

$D(x,y) = -6x(-8) - 0^2 = 48x$

$(2,0): D(2,0) = 48 \cdot 2 = 96 > 0$

$f_{xx}(2,0) = -6 \cdot 2 = -12 < 0$

(2,0) is a rel. max

$(-2,0)$: $D(-2,0) = 48\cdot(-2) = -96 < 0 \rightarrow$ it is a saddle pt

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$f(x,y) = x^3 - y^3 + 6xy$

$f_x = 3x^2 + 6y$ $\quad$ $f_{xx} = 6x$

$f_y = -3y^2 + 6x$ $\quad$ $f_{yy} = -6y$

$f_{xy} = 6$

$$\begin{cases} 3x^2 + 6y = 0 \rightarrow & y = \dfrac{-3x^2}{6} = \dfrac{-x^2}{2} \\ -3y^2 + 6x = 0 & \end{cases}$$

$$-3\left(\frac{-x^2}{2}\right)^2 + 6x = 0$$

$$-3 \cdot \frac{x^4}{4} + 6x = 0$$

$$-\frac{3}{4}x^4 + 6x = 0$$

$$-\frac{3}{4}x\left(x^3 - 8\right) = 0$$

$\underline{x = 0}$ $\quad$ $x^3 = 8$

$\underline{x = 2}$