Section 2.3

$$\frac{d}{dx}\left(x^2(x+1)\right) = \left(x^2(x+1)\right)' = \left(x^3+x^2\right)' = 3x^2+2x$$

Rules for differentiation:

$$(c)' = 0$$
$$(x^n)' = n \cdot x^{n-1}, \quad n \neq 0$$
$$(c \cdot f(x))' = c\,(f(x))'$$
$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

$$(f(x) \cdot g(x))' = f'(x)g(x) + g'(x)f(x)$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$$

Ex: $y = \dfrac{x^2-3}{x}$ Find y'

$$y = \frac{x^2}{x} - \frac{3}{x} = x - 3x^{-1} \qquad y' = (x)' - 3(x^{-1})' = 1 \cdot x^0 - 3(-1)x^{-2}$$
$$= \boxed{1+3x^{-2}} = 1 + \frac{3}{x^2} = \frac{x^2+3}{x^2}$$

$$y = \frac{x^2+5}{x-1} \qquad \text{Find } y'$$

$$y' = \frac{(x^2+5)'(x-1) - (x-1)'(x^2+5)}{(x-1)^2} = \frac{2x(x-1)-(1)(x^2+5)}{(x-1)^2}$$

$$= \frac{2x^2-2x-x^2-5}{(x-1)^2} = \boxed{\frac{x^2-2x-5}{(x-1)^2}}$$

Ex: The polulation of bacteria in a culture is P million where

$$P(t) = \frac{t+1}{t^2+t+4}$$

a) At what rate is the population of the culture changing with respect to time when $t=0$? Is the population increasing or decreasing?

Find $P'(0)$.

$$P'(t) = \frac{(t+1)'(t^2+t+4) - (t^2+t+4)'(t+1)}{(t^2+t+4)^2}$$

$$= \underline{1(t^2+t+4) - (2t+1)(t+1)}$$

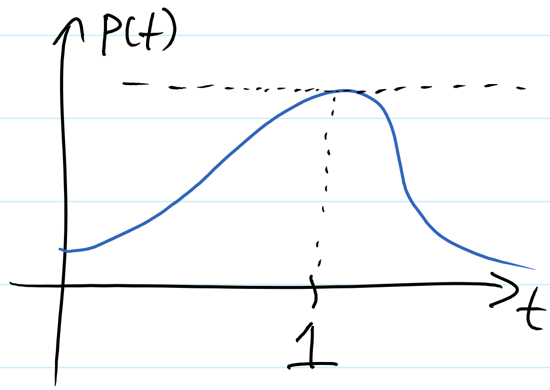
$$(t^2+t+4)^2$$

$$= \frac{t^2+t+4-2t^2-3t-1}{(t^2+t+4)^2} = \frac{-t^2-2t+3}{(t^2+t+4)^2}$$

$$P'(0) = \frac{0-0+3}{4^2} = \boxed{\frac{3}{16}}$$

$3/16 > 0 \Rightarrow$ The population is increasing.

b) At time is the population maximal?
When does it peak?



Find $P'(t)=0$

$$P'(t) = \frac{-t^2-2t+3}{(t^2+t+4)^2} = 0$$

$$-t^2-2t+3=0$$
$$t^2+2t-3=0$$

$$(t-1)(t+3)=0$$

$$\boxed{t=1},\ t=-3$$

<u>Ex:</u> $y=\frac{3}{2x^2}-\frac{x}{3}+\frac{4}{5}+\frac{x+1}{x}$, find $y'=\frac{dy}{dx}$.

$$y=\frac{3}{2}\cdot x^{-2}-\frac{1}{3}x+\frac{4}{5}+\frac{x}{x}+\frac{1}{x}$$

$$=\frac{3}{2}\cdot x^{-2}-\frac{1}{3}x+\frac{4}{5}+1+x^{-1}$$

$$\frac{dy}{dx}=\frac{3}{\cancel{2}}\cdot(-\cancel{2})x^{-3}-\frac{1}{3}\cdot 1+0+0+(-1)x^{-2}$$

$$=\boxed{-3x^{-3}-\frac{1}{3}-x^{-2}}$$

Second derivative:

<u>Def:</u> The second derivative of a function is the derivative of its derivative, i.e.,

$$f''(x)=\frac{d}{dx}\left(\frac{d}{dx}(f(x))\right)=\frac{d^2}{dx^2}(f(x))$$

$$y''=\frac{d^2y}{dx^2}$$

Ex: $y = x^2(3x+1)$ find y''.

$= 3x^3 + x^2$

$y' = 3 \cdot 3x^2 + 2x = 9x^2 + 2x$

$y'' = 9 \cdot 2x + 2 = \boxed{18x+2}$

Ex: The position of an object is given by

$$s(t) = t^3 - 3t^2 + 4t.$$

Find the velocity and the acceleration.

$v(t) = s'(t)$ $\quad\quad a(t) = v'(t) = s''(t)$

$v(t) = 3t^2 - 3 \cdot 2t + 4 = \boxed{3t^2 - 6t + 4}$

$a(t) = 3 \cdot 2t - 6 = \boxed{6t - 6}$

Ex: $f(x) = \frac{1}{x}$, find the fift derivative, $f^{(5)}(x)$

$f(x) = x^{-1}$

$f'(x) = -1 \cdot x^{-2} = -x^{-2}$

$f''(x) = -(-2)x^{-3} = 2x^{-3}$

$f'''(x) = 2 \cdot (-3)x^{-4} = -6x^{-4}$

$f^{(4)}(x) = -6 \cdot (-4)x^{-5} = 24x^{-5}$

$f^{(5)}(x) = 24(-5)x^{-6} = \boxed{-120x^{-6}}$

$\vdots$

$f^{(10)}(x) = (10!) \cdot x^{-11} = (1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot 10)\, x^{-11}$

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① Find the tangent line to $y = (x^2+3)(5-2x^3)$ at $x = 1$

$\boxed{y = -18x + 30}$

② Find $f'(x)$, $f(x) = (x + \frac{1}{x})^2$

③ Find $f'(t)$, $f(t) = 10(3t+1)(1-5t)$

② $f(x) = (x + x^{-1})^2 = x^2 + 2 \cdot \underbrace{x x^{-1}}_{\frac{x}{x}} + x^{-2} = x^2 + 2 + x^{-2}$

$= (x + x^{-1})(x + x^{-1})$

$f'(x) = 2x + 0 + (-2)x^{-3} = \boxed{2x - 2x^{-3}} = \boxed{2x - \frac{2}{x^3}}$

③ $f(t) = 10(3t - 15t^2 + 1 - 5t) = 10(-15t^2 - 2t + 1)$

$= -150t^2 - 20t + 10$

$$f'(t) = -150 \cdot 2t - 20 = \boxed{-300t - 20}$$

Section 2.4 The Chain Rule

How do we differenciate :

$\sqrt{x} = x^{1/2}$ ✓ | $\sqrt{x+1}$ × | $(x+3)^{20}$ ×

$(x^5 - 3x^3)^4$ × | $\dfrac{\sqrt{x-3}}{x+5}$ ×

$y = x^3$ the derivative of y is $\frac{dy}{dx} = 3x^2$

$$y(x) = (2x-3)^2 = 4x^2 - 12x + 9$$

$$y'(x) = 4 \cdot 2x - 12 = 8x - 12$$

$$y(x) = (u(x))^2 \quad , \quad u(x) = 2x-3$$

$$y = u^2 \quad , \quad u = 2x - 3$$

$y = u^2$, $u = 2x - 3$

$y'(u) = 2u$, $u'(x) = 2$

Find $y'(x) = y' = \frac{dy}{dx} \cdot \frac{du}{du} = \frac{dy}{du} \cdot \frac{du}{dx} = y'(u) \cdot u'(x)$

$= 2u \cdot 2$

$= 2 \cdot (2x-3) \cdot 2$

$= 4(2x-3)$

$= \boxed{8x - 12}$

$y = \frac{3x^2 - 1}{3x^2 - 1 + 1}$, $u = 3x^2 - 1$, Find y'

$y = \frac{u}{u+1}$ | $y' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = y'(u)\, u'$

$= \left(\frac{u}{u+1}\right)' \cdot \left(3x^2 - 1\right)'$

$= \frac{(u)'(u+1) - (u+1)' \cdot u}{(u+1)^2} \cdot (6x)$

$= \frac{\cancel{u} + 1 - \cancel{u}}{(u+1)^2} \cdot 6x = \frac{1}{(u+1)^2} \cdot 6x$

$= \frac{6x}{(3x^2 - 1 + 1)^2} = \frac{6x}{(3x^2)^2} = \frac{6x}{9x^4} = \boxed{\frac{2}{3x^3}}$

$y = \frac{3x^2 - 1}{3x^2 - 1 + 1} = \frac{3x^2 - 1}{3x^2} = \frac{3x^2}{3x^2} - \frac{1}{3x^2} = 1 - \frac{1}{3}x^{-2}$

$$y' = 0 - \frac{1}{3}(-2)x^{-3} = \frac{2}{3}x^{-3} = \boxed{\frac{2}{3x^3}}$$

Def: The derivative of $f(g(x))$,

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

$$\left(y(u)\right)' = y'(u) \cdot u'(x)$$

Ex: $y = \sqrt{x+1}$

$u = x+1 \rightarrow u'(x) = 1+0 = 1$

$y = \sqrt{u} \rightarrow y'(u) = \frac{1}{2}u^{-1/2}$

$$\frac{dy}{dx} = y'(u) \cdot u'(x) = \frac{1}{2}u^{-1/2} \cdot 1 = \frac{1}{2}(x+1)^{-1/2} = \boxed{\frac{1}{2\sqrt{x+1}}}$$