MAC 2233, Fall 2017

Exam #4

December 7, 2017

Name ____________________________

- You will be told when to begin the work and when to terminate work on the examination. You must stop when instructed. Points may be deducted in case of violations.
- Please show your work to support your answers that require calculations. Correct but unsupported answers may not be given full credit.
- The use of a cell phone or other electronic communication devices during the examination is not allowed. The exam will be canceled and a grade of "0" will be assigned to anyone who opens a cell phone during the examination or if one is found on their seat or hand.

No calculators are allowed!

$$\boxed{D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - \Big(f_{xy}(x,y)\Big)^2}$$

Honor Code: On my honor, I have neither received nor given any aid during this examination.

Signature: ________________________

1. (12.5 pts each) Find the second partial derivatives (mixed derivative included)

(a) $f(x,y) = e^{xy}$

$f_x = e^{xy}\cdot y = y e^{xy}$

$f_y = e^{xy}\cdot x = x e^{xy}$

$f_{xx} = y e^{xy}\cdot y = \boxed{y^2 e^{xy}}$

$f_{yy} = x e^{xy}\cdot x = \boxed{x^2 e^{xy}}$

$f_{xy} = 1\cdot e^{xy} + y\cdot e^{xy}\cdot x$

$= \boxed{e^{xy} + xy e^{xy}}$

$= \boxed{e^{xy}(1+xy)}$

(b) $f(x,y) = \ln(x^2 + y)$

$f_x = \frac{2x}{x^2+y} = 2x(x^2+y)^{-1}$

$f_y = \frac{1}{x^2+y} = (x^2+y)^{-1}$

$f_{xx} = 2(x^2+y)^{-1} + 2x(-1)(x^2+y)^{-2}\cdot 2x$

$= \boxed{2(x^2+y)^{-1} - 4x^2(x^2+y)^{-2}}$

$f_{yy} = -(x^2+y)^{-2}\cdot 1 = \boxed{-(x^2+y)^{-2}}$

$f_{yx} = -(x^2+y)^{-2}\cdot 2x = \boxed{-2x(x^2+y)^{-2}}$

2. (10 pts) The demand function for peanut butter is

$$D_1(p_1, p_2) = 800 - 3p_1 - 4p_2$$

while that for a second commodity is

$$D_2(p_1, p_2) = 500 - 2p_2^3 - \frac{p_1}{2}$$

Is the second commodity more likely to be jelly or bread? Explain. [Hint: are the two commodities substitute or complementary?]

$$\left.\begin{aligned} \frac{\partial D_1}{\partial p_2} &= 0-0-4 = -4 < 0 \\ \frac{\partial D_2}{\partial p_1} &= 0-0-\tfrac{1}{2} = -\tfrac{1}{2} < 0 \end{aligned}\right\}$$

The commodities are complementary. Hence the second comm. is probably bread.

3. (10 pts) Determine whether the commodities are substitute, complementary, or neither.

$$\begin{aligned} D_1 &= 1000 - \frac{300}{p_1+3} + 50p_2 \\ D_2 &= 2000 - 80p_1 + \frac{300}{p_2+4} \end{aligned}$$

$$\left.\begin{aligned} \frac{\partial D_1}{\partial p_2} &= 50 > 0 \\ \frac{\partial D_2}{\partial p_1} &= -80 < 0 \end{aligned}\right\} \underline{\text{neither}}$$

4. (10 pts) A grocer's daily profit from the sale of two brands of cat food is

$$P(x, y) = 5x^2 + 2xy - 460x - 7y^2 + 480y - 1100$$

dollars, where x is the price per can of the first brand and y is the price per can of the second. Currently the first brand sells for \$2 per can and the second for \$5 per can.

(a) Use marginal analysis to estimate the change in daily profit that will result if the grocer raises the price of the **first** brand by \$1 per can but keeps the price of the second brand unchanged.

$$\frac{\partial P}{\partial x} = 10x + 2y - 460$$

$$\frac{\partial P}{\partial x}(2,5) = 10\cdot 2 + 2\cdot 5 - 460 = 30 - 460 = \underline{-430}$$

The daily profit decreases by \$430.

(b) Use marginal analysis to estimate the change in daily profit that will result if the grocer raises the price of the **second** brand by \$1 per can but keeps the price of the first brand unchanged.

$$\frac{\partial P}{\partial y} = 2x - 14y + 480$$

$$\frac{\partial P}{\partial y}(2,5) = 2\cdot 2 - 14\cdot 5 + 480 = 4 - 70 + 480$$

$$= \underline{414}$$

The daily profit increases by \$414.

(c) Increasing the price of which brand will yield higher daily profit?

The second brand.

5. (15 pts each) Find the critical points of the given function and classify each as a relative maximum, a relative minimum, or a saddle point.

(a) $f(x, y) = xy$

$f_x = y$

$f_y = x$

$f_{xx} = 0$

$f_{yy} = 0$

$f_{xy} = 1$

Crit. pts: $\begin{cases} y = 0 \\ x = 0 \end{cases}$ $(0,0)$

$D = 0 \cdot 0 - 1^2 = -1$

$D(0,0) = -1 < 0 \to$ $(0,0)$ is a saddle point.

(b) $f(x, y) = x^3 - 12x + y^2 + 2y + 2$

$f_x = 3x^2 - 12$

$f_y = 2y + 2$

$f_{xx} = 6x$

$f_{yy} = 2$

$f_{xy} = 0$

Crit. pts:

$\begin{cases} 3(x^2 - 4) = 0 \to x^2 = 4, \ x = \pm 2 \\ 2(y+1) = 0 \to y = -1 \end{cases}$ $(2,-1)$, $(2,+1)$

$D = 6x \cdot 2 - 0^2 = 12x$

$(2,-1)$: $D(2,-1) = 12 \cdot 2 = 24 > 0$, $f_{xx}(2,-1) = 6 \cdot 2 = 12 > 0$ $\to$ $(2,-1)$ is a relative minimum.

$(-2,-1)$: $D(-2,-1) = 12 \cdot (-2) = -24 < 0 \to$ $(-2,-1)$ is a saddle point

(c) $f(x, y) = 4xy - x^2 - 4y + 9$

$f_x = 4y - 2x = 0 \to 4y - 2 \cdot 1 = 0 \to y = \frac{2}{4}$, $y = \frac{1}{2}$

$f_y = 4x - 4 = 0 \to x = 1$

$\boxed{(1, \tfrac{1}{2})}$

$f_{xx} = -2$

$f_{yy} = 0$

$f_{xy} = 4$

$D = -2 \cdot 0 - 4 = -4$

$D(1, \frac{1}{2}) = -4 < 0 \to$ $(1, 1/2)$ is a saddle point

5. (5 extra credit points) Find the critical points of the functions in problem 1 and classify each as a relative maximum, a relative minimum, or a saddle point.

(a) $f(x, y) = e^{xy}$

Crit. pts: $\begin{cases} y e^{xy} = 0 \to y = 0 \\ x e^{xy} = 0 \to x = 0 \end{cases}$ $(0,0)$

$D = y^2 e^{xy} \cdot x^2 e^{xy} - (e^{xy}(1+xy))^2$

$D(0,0) = 0 \cdot e^0 \cdot 0 \cdot e^0 - (1(1+0))^2 = 0 - 1^2 = -1 < 0$

$\boxed{(0,0) \text{ is a saddle pt}}$

(b) $f(x, y) = \ln(x^2 + y)$

Crit. pts: $\begin{cases} \frac{2x}{x^2+y} = 0 \\ \frac{1}{x^2+y} = 0 \end{cases}$ ← doesn't have a solution, i.e., $f_y \neq 0 \Rightarrow$ $\boxed{\text{no critical points}}$

6. (2 extra credit points each) Determine if the following statement is true or false.

(a) (true / false)
A function $f(x, y)$ has a relative maximum at (a, b) if the sign of both first derivatives change from positive to negative about the point (a, b).

(b) (true/false)
If a function $f(x, y)$ has all first and second partial derivatives $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$ then $f_{xy} = f_{yx}$.