MTH130, Spring 2017

Final Exam

May 3, 2017

Name ____________________________

- You will be told when to begin the work and when to terminate work on the examination. You must stop when instructed. Points may be deducted in case of violations.
- Please show your work to support your answers that require calculations. Correct but unsupported answers may not be given full credit.
- The use of a cell phone or other electronic communication devices during the examination is not allowed. The exam will be canceled and a grade of "0" will be assigned to anyone who opens a cell phone during the examination or if one is found on their seat or hand.

No graphing calculators are allowed!!

Average cost function:	$\overline{C}(x) = \dfrac{C(x)}{x}$
Revenue function:	$R(x) = p * x$
Profit function:	$P(x) = R(x) - C(x)$
Elasticity of demand:	$E(p) = -\dfrac{pf'(p)}{f(p)}$
Differential:	$\mathrm{d}y = f'(x)\ \mathrm{d}x$
Average value:	$\frac{1}{b-a} \int_a^b f(x)\ \mathrm{d}x$

1. (5 points each) Evaluate the limits algebraically, if they exist

a) $\lim_{x\to 3} \frac{2x(x-3)}{\sqrt{x^2-2x}}$ $\longrightarrow$ $\frac{2\cdot 3(3-3)}{\sqrt{9-6}} = \frac{6\cdot 0}{\sqrt{3}} = \frac{0}{\sqrt{3}} = 0$

$= \boxed{0}$

b) $\lim_{x\to 5^+} \frac{x-6}{\sqrt{x-1}-1}$ $\longrightarrow$ $\frac{5-6}{\sqrt{5-1}-1} = \frac{-1}{\sqrt{4}-1} = \frac{-1}{2-1} = \frac{-1}{1} = -1$

$= \boxed{-1}$

$\frac{0}{0} \to$ do algebra

$\frac{\#}{0} \to$ one sided limits

c) $\lim_{x\to\infty} \frac{3x+102}{4x^2+2x-1}$

$\lim_{x\to\infty} \frac{3x+102}{4x^2+2x-1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\cancel{\frac{3}{x}}^{\to 0} + \cancel{\frac{102}{x^2}}^{\to 0}}{4 + \cancel{\frac{2}{x}}_{0} - \cancel{\frac{1}{x^2}}_{0}} = \frac{0}{4} = \boxed{0}$

d) $\lim_{x\to\infty} \frac{3x^3+2x-4}{x^2-x} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3x + \cancel{\frac{2}{x}}^{\to 0} - \cancel{\frac{4}{x^2}}^{\to 0}}{1 - \cancel{\frac{1}{x}}_{0}} = \frac{3\cdot\infty}{1} = \boxed{\infty}$

2. (5 points) Find the derivative of the function using the **definition of derivative**. Find an equation of the tangent line at the point $x = 2$.

$$f(x) = x^2 - 2x \longrightarrow f(2) = 4 - 4 = 0$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - 2(x+h) - (x^2 - 2x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h - \cancel{x^2} + \cancel{2x}}{h} = \lim_{h\to 0} 2x + h - 2 = \underline{2x - 2}$$

$$f'(2) = 4 - 2 = \underline{2} \qquad \underset{\substack{\| \\ 0}}{y - y_0} = 2(x - \underset{\substack{\| \\ 2}}{x_0}) \longrightarrow \boxed{y = 2(x-2)}$$

3. (5 points) The demand equation for a certain product is $p = 25 - 0.1x$, where p is the unit price and x is the quantity demanded of the product.

(a) Find the marginal revenue function, $R'(x)$, and compute its value at $x = 2$.

$$R(x) = p \cdot x = (25 - 0.1x)x = 25x - 0.1x^2$$

$$R'(x) = \boxed{25 - 0.2x}$$

(b) Use the equation $x = f(p) = 250 - 10p$ to find the formula for the elasticity of demand. Is the demand elastic, unitary or inelastic when $p = 15$?

$$E = \frac{-p\,q'(p)}{q(p)} = \frac{-p \cdot (-10)}{250 - 10p} = \frac{10p}{250 - 10p}$$

$$= \boxed{\frac{p}{25 - p}} \xrightarrow{p=15} \frac{15}{25 - 15} = \frac{15}{10} = 1.5 > 1$$

$\boxed{\text{Elastic}}$

4. (5 points) Acrosonic's production department estimates that the total cost (in dollars) incurred in manufacturing x ElectroStat speaker systems in the first year of production will be represented by the following function, where $R(x)$ is the revenue function in dollars and x denotes the quantity demanded.

$$C(x) = 300x + 40000 \quad \text{and} \quad R(x) = -0.04x^2 + 800x$$

(a) Find the profit function $P(x)$

$$P(x) = -0.04x^2 + 800x - 300x - 40000$$
$$= \boxed{-0.04x^2 + 500x - 40000}$$

(b) Find the marginal profit function $P'(x)$

$$P'(x) = \boxed{-0.08x + 500}$$

(c) What is the marginal profit when $x = 3600$?

$$P'(3600) = \boxed{-0.08 \cdot 3600 + 500}$$

(d) What is the actual profit in producing the 3601st speaker system?

$$P(3601) - P(3600) = \ldots$$

5. (5 points) ~~A particle moves along a line so that its position at time t is $s(t) = \frac{1}{12}t^4 - \frac{1}{2}t^3 + t^2 - 3t + 8$. Find the acceleration function $a(t)$ and all times t at which the particle does not accelerate, i.e, $a(t) = 0$.~~

6. (5 points each) Find the derivative of the function

(a) $f(x) = e^{x^3-2x+12}$

$$f'(x) = \boxed{e^{x^3-2x+12} \cdot (3x^2-2)}$$

(b) $g(x) = \ln(-2x^2 + x)$

$$g'(x) = \frac{1}{-2x^2+x} \cdot (-4x+1) = \boxed{\frac{-4x+1}{-2x^2+x}}$$

$\ln x^2 = 2\ln x$

(c) $h(x) = x\ln(x^2)$

$$h'(x) = (x)' \cdot \ln(x^2) + (\ln(x^2))' \cdot x$$
$$= 1 \cdot 2\ln x + (2\ln x)' \cdot x$$
$$= 2\ln x + 2 \cdot \frac{1}{\cancel{x}} \cdot \cancel{x} = \boxed{2\ln x + 2}$$

7. (5 points) Find the absolute maximum and minimum value of the function $f(x) = 2 + (x-2)^2$ on the interval $[-2, 5]$.

$f'(x) = 0 + 2(x-2) = 0$

$\underline{x = 2}$

x	$f(x)$
2	$2+(2-2)^2 = 2$ ← abs. min
-2	$2+(-4)^2 = 18$ ← abs. max
5	$2+(3)^2 = 11$

8. (10 points) Consider the function $f(x) = x^4 - 2x^3 + 2$.

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the local min/max of f.

(c) Find the intervals of concavity and the inflection points.

$f'(x) = 4x^3 - 6x^2$

$f'' = 12x^2 - 12x$

min/max: $4x^3 - 6x^2 = 0$

$2x^2(2x-3) = 0$

$x = 0, 3/2$

f'	$(-\infty, 0)$	$(0, 3/2)$	$(3/2, \infty)$
$2x^2$	+	+	+
$(2x-3)$	−	−	+
	−	−	+
	↘	↘	↗

$\boxed{x = \frac{3}{2} \text{ is a rel. min}}$

$12x^2 - 12x = 0$

$12x(x-1) = 0$

$x = 0, 1$

f''	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$12x$	−	+	+
$x-1$	−	−	+
	+	−	+
	∪	∩	∪

$\boxed{x = 0, 1 \text{ are inflection pts}}$

9. (5 points) Solve only **one** of the following problems:

(a) Your car will need new tires in 2 years. Assume that the price for 4 tires with installation is \$500. Determine how much you have to deposit in your savings account today to save for this expense if your savings account is compounded continuously with 5% interest.

(b) Your bike will need new tires in 2 years. Assume that the price for 2 tires without installation is \$50. Determine how much you have to deposit in your savings account today to save for this expense if your savings account is compounded monthly with 5% interest.

a) $B(t) = Pe^{rt}$

$$\frac{500}{e^{0.1}} = \frac{P \cdot e^{0.05 \cdot 2}}{e^{0.1}}$$

$$P = \boxed{\frac{500}{e^{0.1}}}$$

b) $B(t) = P\left(1+\frac{k}{r}\right)^{kt}$

$$\frac{50}{\left(1+\frac{12}{0.05}\right)^{24}} = \frac{P\left(1+\frac{12}{0.05}\right)^{12\cdot 2}}{\left(1+\frac{12}{0.05}\right)^{24}}$$

$$P = \boxed{50\left(1+\frac{12}{0.05}\right)^{-24}}$$

10. (5 points) Find the relative extrema, if any, of the function

$$f(x) = \frac{2}{1-x^2} = 2(1-x^2)^{-1}$$

$$f'(x) = 2\cdot(-1)(1-x^2)^{-2}\cdot(-2x) = 4x(1-x^2)^{-2}$$

$$= \frac{4x}{(1-x^2)^2}$$

$4x = 0$
$x = 0$

$1-x^2=0$
$1 = x^2$
$x = \pm 1$

f'	$(-\infty, -1)$	$(-1, 0)$	$(0,1)$	$(1,\infty)$
$4x$	$-$	$-$	$+$	$+$
$(1-x^2)^2$	$+$	$+$	$+$	$+$
	$-$	$-$	$+$	$+$
	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

$\underline{x=0}$ is a rel. min

11. (5 points) Check that $F(x)$ is an antiderivative of $f(x)$

(a) $F(x) = \frac{-1}{x} - \frac{\ln x}{x} + 1; \quad f(x) = \frac{\ln(x)}{x^2}$

$$\frac{d}{dx}\left(F(x)\right) = \left(\frac{-1-\ln x}{x} + 1\right)' = \frac{\left(-\frac{1}{x}\right)\cdot x - (-1-\ln x)\cdot 1}{x^2}$$

$$= \frac{\cancel{-1}\ \cancel{+1} + \ln x}{x^2} = \frac{\ln x}{x^2} \checkmark$$

(b) $F(x) = 3 - \frac{x^2}{4} + \frac{1}{2}x^2\ln(x); \quad f(x) = x\ln(x)$

12. (5 points each) Find the general indefinite integral.

(a)

$$\int \frac{\sqrt{x} + 4x^2}{x}\,dx = \int \frac{x^{1/2}}{x} + 4\frac{x^2}{x}\,dx$$

$$= \int x^{-1/2} + 4x\,dx$$

$$= \frac{2}{1}x^{1/2} + 4\frac{1}{2}x^2 + C$$

$$= \boxed{2\sqrt{x} + 2x^2 + C}$$

(b)

$$\int t - \frac{1}{t^4}\, dt = \int t - t^{-4}\, dt$$

$$= \frac{1}{2}t^2 - \frac{1}{-3}t^{-3} + C$$

$$= \boxed{\frac{1}{2}t^2 + \frac{1}{3}t^{-3} + C}$$

(c)

$$\int 2xe^{x^2}\, dx$$

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$= \int \cancel{2}\, e^u\, \cancel{\tfrac{1}{2}}\, du = \int e^u\, du = e^u + C$$

$$= \boxed{e^{x^2} + C}$$

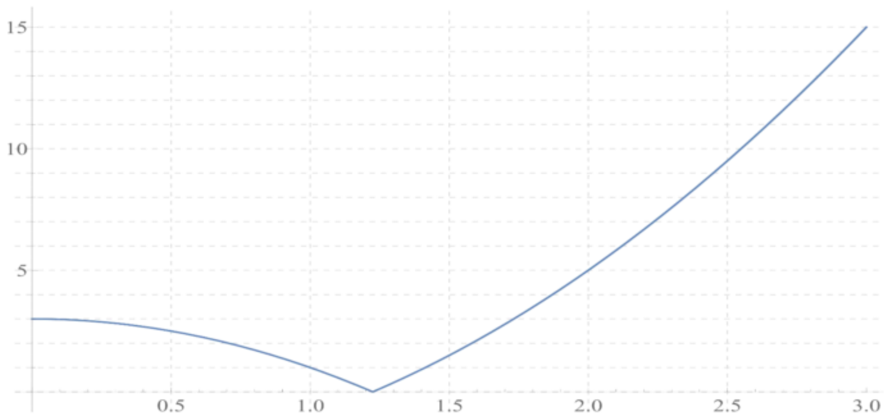
13. (5 points) Find the average value of the function $f(x) = \sqrt{x}$ on the interval $[4, 9]$. Simplify your answer.

$$\frac{1}{9-4}\int_4^9 \sqrt{x}\, dx = \frac{1}{5}\int_4^9 x^{1/2}\, dx = \frac{1}{5}\cdot\frac{2}{3}x^{3/2}\Big|_4^9$$

$$= \frac{2}{15}\left(9^{3/2} - 4^{3/2}\right) = \frac{2}{15}\left(3^3 - 2^3\right)$$

$$= \frac{2}{15}(27 - 8) = \frac{2\cdot 19}{15} = \boxed{\frac{38}{15}}$$

14. (5 points) ~~Estimate the area under the graph of $f(x) = |2x^2 - 3|$ from $x = 0$ to $x = 3$ using three approximating rectangles and left endpoints, draw the approximating rectangles. Simplify your answer.~~



15. (5 points each) Evaluate the integrals, simplify your answer

(a)

$$\int_0^1 x^2(x^3+2)^2\,dx \quad = \quad \begin{array}{|l} u = x^3+2 \\ du = 3x^2\,dx \\ \frac{1}{3}du = x^2\,dx \end{array}$$

$$= \frac{1}{3}\int u^2\,du = \frac{1}{3}\,\frac{1}{3}u^3\Big| = \frac{1}{9}(x^3+2)^3\Big|_0^1 = \frac{1}{9}(1+2)^3 - \frac{1}{9}(0+2)^3$$

$$= \frac{3^3}{9} - \frac{2^3}{9} = \boxed{\frac{19}{9}}$$

(b)

$$\int_{-1}^{2} \frac{x^2-1}{x-1}\,dx = \int_{-1}^{2} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}\,dx = \int_{-1}^{2} x+1\,dx$$

$$= \frac{1}{2}x^2+x\Big|_{-1}^{2} = \frac{1}{2}\cdot 4+2-\left(\frac{1}{2}-1\right)$$

$$= 2+2-\left(-\frac{1}{2}\right) = 4+1/2 = \boxed{4.5}$$

$\ln x^3 = 3\ln x$

(c)

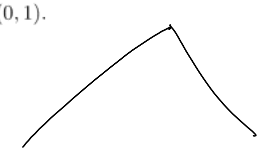
$$\int_{1}^{e} \frac{(\ln x)^3}{x}\,dx \quad \Bigg| \begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array}$$

$$= \int u^3\,du = \frac{1}{4}u^4 \Big| = \frac{1}{4}(\ln x)^4\Big|_{1}^{e}$$

$$= \frac{1}{4}(\ln e)^4 - \frac{1}{4}(\ln 1)^4 = \frac{1}{4}\cdot 1 - \frac{1}{4}\cdot 0 = \boxed{\frac{1}{4}}$$

16. (**2 extra points each**) No justification necessary.

(a) (True/False) If f is continuous on $[0, 1]$, then f is differentiable on $(0, 1)$.



(b) (True/False) Given a continuous function $f(x)$ and its antiderivative $F(x)$, the following identity holds for all constants a and b.

$$\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a)$$

17. (**3 extra points each**) Evaluate the integrals and simplify your answers. [Hint: Do not use substitution method to solve the integrals]

(a)

$$\int_1^{e^2} \frac{\ln x}{x^2}\ \mathrm{d}x = \frac{-1}{x} - \frac{\ln x}{x} + 1 \Big|_1^{e^2}$$

$$= \frac{-1}{e^2} - \frac{\ln e^2}{e^2} \cancel{+1} - \left(\frac{-1}{1} - \frac{\overbrace{\ln 1}}{\underbrace{1}_{0}} \cancel{+1}\right) = \boxed{\frac{-1}{e^2} - \frac{2}{e^2} + 1}$$

(b)

$$\int_1^{e} x \ln(x)\ \mathrm{d}x = \cancel{3} - \frac{x^2}{4} + \frac{1}{2}x^2 \ln x \Big|_1^{e}$$

$$\frac{-e^2}{4} + \frac{1}{2}e^2 \ln e - \left(\frac{1}{4} + \frac{1}{2}\cdot 1^2 \cdot \underbrace{\ln 1}_{0}\right) = \boxed{\frac{-e^2}{4} + \frac{1}{2}e^2 - \frac{1}{4}}$$

Honor Code: *On my honor, I have neither received nor given any aid during this examination.*

Signature: ____________________