

#5 (a)  $\text{div}(x, y)$  was defined to be the largest integer  $n$  such that  $x \geq ny$ . When  $y=0$ ,  $\text{div}(x, y)$  is undefined because there is no largest integer  $n$  such that  $x \geq n \cdot 0$ . So we will amend the definition as follows:

$$\text{div}(x, y) = \begin{cases} 0 & \text{if } y=0 \\ \text{largest } n \text{ such that } x \geq ny & \text{if } y \neq 0. \end{cases}$$

Now  $\text{div}(0, y) = 0$

$$\text{div}(x+1, y) = \begin{cases} \text{div}(x, y) & \text{if } x+1 < y(1 + \text{div}(x, y)) \\ \text{div}(x, y) + 1 & \text{if } x+1 = y(1 + \text{div}(x, y)) \end{cases}$$

$\therefore \text{div}(0, y) = 0$

$$\text{div}(x+1, y) = \text{div}(x, y) + \text{equals}(x+1, y + (y \cdot \text{div}(x, y)))$$

Now this is almost okay except the recursion is being done on the first coordinate. In primitive recursion, we have to do it on the last coordinate.

$$\text{div}(x, y) = f(y, x) = f(p_2(x, y), p_1(x, y))$$

where  $f(x, 0) = z(x)$

$$\begin{aligned} f(x, y+1) &= \text{div}(y+1, x) \\ &= \text{div}(y, x) + \text{equals}(y+1, x + x \cdot \text{div}(y, x)) \\ &= f(x, y) + \text{equals}(y+1, x + x \cdot f(x, y)) \\ &= h(x, y, f(x, y)) \end{aligned}$$

where

$$h(x, y, z) = \text{add}(p_3(x, y, z), \text{equals}(s(p_2(x, y, z)), \text{add}(p_1(x, y, z), \text{mult}(p_1(x, y, z), p_3(x, y, z))))$$

$\therefore \text{div}(x, y)$  is primitive recursive.

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$$5 \text{ (b) } \text{rem}(x, y) = x \dot{-} y \cdot \text{div}(x, y) \\ = \text{monus}(p_1(x, y), \text{mult}(p_2(x, y), \text{div}(x, y)))$$

$$\text{(c) } \text{max}(x, y) = x + y \dot{-} x \\ = \text{add}(p_1(x, y), \text{monus}(p_2(x, y), p_1(x, y)))$$

$$\text{(d) } \text{min}(x, y) = x \dot{-} (x \dot{-} y) \\ = \text{monus}(p_1(x, y), \text{monus}(p_1(x, y), p_2(x, y)))$$

$$\text{(e) } \text{gr}(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases} \\ = 1 \dot{-} \text{equal}(x \dot{-} y, 0)$$

$$= \text{monus}(c_1(p_1(x, y), \text{equal}(\text{monus}(x, y), z(p_1(x, y))))$$

$$\text{(f) } f(x) = \lfloor \sqrt{x} \rfloor = (\mu y) ((y+1)^2 > x) \\ = (\mu y) [g(x, y) = 0],$$

$$\text{where } g(x, y) = 1 \dot{-} \text{gr}((y+1)^2, x) = \\ \text{monus}(c_1(p_1(x, y), \text{gr}(\text{mult}(s(p_2(x, y), s(p_2(x, y))), p_1(x, y))))$$

$$\text{(g) } f(x) = \lfloor \log_2(x+1) \rfloor = (\mu y) [2^{y+1} > x+1] \\ = (\mu y) [g(x, y) = 0] \text{ where}$$

$$g(x, y) = 1 \dot{-} \text{gr}(2^{y+1}, x+1) \\ = \text{monus}(c_1(p_1(x, y), \text{gr}(\text{exp}(c_2 p_1(x, y), s(p_2(x, y))), s(p_2(x, y))))$$

$$\begin{aligned}
 9 \text{ (a)} \quad A(1, y) &= A(1, (y-1)+1) \\
 &= A(1-1, A(1, y-1)) \\
 &= A(0, A(1, y-1)) \\
 &= A(1, y-1) + 1 \\
 &= A(1, (y-2)+1) + 1 \\
 &= A(0, A(1, y-2)) + 1 \\
 &= A(1, y-2) + 1 + 1 \\
 &\quad \vdots \\
 &= A(1, y-y) + \underbrace{1 + 1 + \dots + 1}_{y \text{ times}} \\
 &= A(1, 0) + y \\
 &= A(0, 1) + y \\
 &= (1+1) + y \\
 &= y + 2.
 \end{aligned}$$

$$6. \quad f(0) = 2^0 = 1$$

$$f(n+1) = 2^{n+1} = 2^n + 2^n = \text{Add}(f(n), f(n))$$

From this we can see that  $f(n)$  is prim. recursive

$$f = \text{prim. rec.} (s_0, \text{ADD} \circ [I_2^{(2)}, I_2^{(2)}])$$

$$7. \quad g(x, 0) = x^0 = 1$$

$$g(x, y+1) = x^{y+1} = x^y \cdot x$$

From this we can see that  $g(x, y) = x^y$  is prim. recursive

$$g = \text{prim. rec.} (s_0, \text{MULT} \circ [I_3^{(3)}, I_1^{(3)}])$$

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9 (a)  $A(1, y) = y + 2$  (done on page 40)

(b)  $A(2, y) = A(2, (y-1)+1) = A(2-1, A(2, y-1))$   
 $= A(1, A(2, y-1)) = A(2, y-1) + 2$  by (a)  
 $= A(2, (y-2)+1) + 2 = A(2-1, A(2, y-2)) + 2$   
 $= A(1, A(2, y-2)) + 2 = (A(2, y-2) + 2) + 2$  by (a)  
 $= \dots$   
 $= A(2, y-y) + \underbrace{2+2+\dots+2}_{y \text{ times}}$   
 $= A(2, 0) + 2y = A(1, 1) + 2y$   
 $= 1 + 2 + 2y$  by (a)  
 $= 2y + 3$

(c)  $A(3, y) = A(3, (y-1)+1) = A(3-1, A(3, y-1))$   
 $= A(2, A(3, y-1)) = 2A(3, y-1) + 3$  by (b)  
 $= 2[A(3, y-1) + 3] - 3$   
 $= 2[A(2, A(3, y-2)) + 3] - 3$   
 $= 2 \cdot [2A(3, y-2) + 3 + 3] - 3$  by (b)  
 $= 2 \cdot 2 \cdot [A(3, y-2) + 3] - 3$   
 $= \dots$   
 $= \underbrace{2 \cdot 2 \cdot 2 \dots 2}_{y \text{ times}} \cdot [A(3, y-y) + 3] - 3$   
 $= 2^y \cdot [A(3, 0) + 3] - 3$   
 $= 2^y \cdot [A(2, 1) + 3] - 3$   
 $= 2^y \cdot [(2 \cdot 1 + 3) + 3] - 3$   
 $= (2^y \cdot 8) - 3 = 2^{y+3} - 3$

$$\begin{aligned}
 10. (a) \quad A(4,1) &= A(4, 0+1) = A(3, (A(4,0))) \\
 &= 2^{A(4,0)+3} - 3 \quad \text{by 9(c)} \\
 &= 2^{A(3,1)+3} - 3 \\
 &= 2^{2^{1+3}-3+3} - 3 \quad \text{by 9(c)} \\
 &= 2^{18} - 3
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad A(4,2) &= A(4, 1+1) = A(3, A(4,1)) \\
 &= 2^{A(4,1)+3} - 3 \\
 &= 2^{2^{18}-3+3} - 3 \quad \text{by 10(a)} \\
 &= 2^{2^{16}} - 3
 \end{aligned}$$

11. In general

$$A(7,4) = \underbrace{2^{2^{\dots^{2^2}}}}_{y \text{ times}} - 3 = \underbrace{2^{2^{\dots^{2^2}}}}_{y+3 \text{ times}} - 3.$$

15 (a)  $(\mu y) [g(x, y) = 0]$   
 = smallest  $y$  such that  $x \cdot y = 0$   
 = 0 because  $x \cdot 0 = 0$  & 0 is the smallest  
 value of  $y$  with  $x \cdot y = 0$ .  
 domain =  $\mathbb{N}$ .

(b)  $(\mu y) [2^x + y - 3 = 0]$   
 = smallest  $y$  such that  $2^x + y - 3 = 0$   
 =  $\begin{cases} 3 - 2^0 & \text{if } x=0 \text{ bec. } 2^0 + 0 - 3 \neq 0 \& 2^0 + 1 - 3 \neq 0 \\ 3 - 2^1 & \text{if } x=1 \text{ bec. } 2^1 + 0 - 3 \neq 0 \\ \text{undefined} & \text{if } x > 1 \end{cases}$   
 =  $\begin{cases} 2 & \text{if } x=0 \\ 1 & \text{if } x=1 \\ \text{und} & \text{if } x > 1 \end{cases}$

domain =  $\{0, 1\}$ . (The textbook is sloppy - it should not use "-".  
 It should use " $\dot{=}$ " (monus) but it didn't.)

(c)  $(\mu y) \left( \lfloor \frac{x-1}{y+1} \rfloor = 0 \right)$   
 =  $\begin{cases} \text{und.} & \text{if } x=0 & \text{because } \lfloor \frac{-1}{y+1} \rfloor = -1 \text{ for all } y \geq 0 \\ 0 & \text{if } x=1 & \text{because } \lfloor \frac{0}{y+1} \rfloor = 0 \\ x-1 & \text{if } x \geq 2 & \text{because } \lfloor \frac{x-1}{y+1} \rfloor \neq 0 \text{ for all } y \leq x-2 \\ & & \text{and } \lfloor \frac{x-1}{x-1+1} \rfloor = 0 \text{ for all } x \geq 2 \end{cases}$   
 =  $\begin{cases} \text{und} & \text{if } x=0 \\ x-1 & \text{if } x > 0. \end{cases}$   
 domain =  $\mathbb{N} - \{0\}$

(d)  $(\mu y) (x \text{ [mod}(y+1)] = 0) = \begin{cases} 0 & \text{if } x=0 \text{ bec. } 0 \text{ (mod } 1) = 0 \\ 0 & \text{if } x > 0 \text{ bec. } x \text{ (mod } 1) = 0 \end{cases}$   
 = 0 if  $x \in \mathbb{N}$   
 domain =  $\mathbb{N}$ . (A rather silly question) END