

MAD 3512 - THEORY OF ALGORITHMS
TEST #2 – SP 2023

FLORIDA INTL UNIV.
TIME: 75 min.

Answer all 6 questions. No calculators, notes, or on-line stuff are allowed. An unjustified answer will receive little or no credit. Draw a line to separate each of your 6 solutions to the 6 questions.

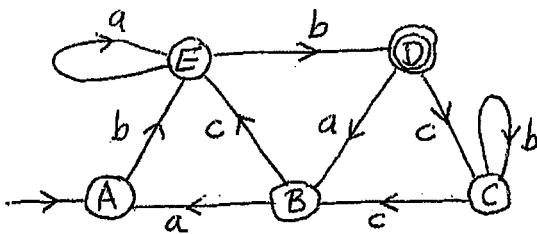
- (16) 1. (a) Find an NFA, M , which is *equivalent* to the RLG G given below.

$$G: \begin{array}{l} \rightarrow B, \quad B \rightarrow 10B, \quad B \rightarrow 1C, \quad C \rightarrow 10, \quad C \rightarrow \lambda, \quad C \rightarrow 1D, \\ \quad C \rightarrow E, \quad D \rightarrow 0B, \quad E \rightarrow \lambda, \quad E \rightarrow 1C, \quad E \rightarrow 00. \end{array}$$

- (b) Find an RLG, G , which is equivalent to the NFA in Problem 2(a) below.

- (16) 2. (a) Find a *regular expression* for the language accepted by the NFA M shown on the right.

- (b) Define what is the *Busy-beaver function*.



- (16) 3. (a) Write down the *initial functions* & define what " $F = \text{prec}(g,h)$ " means.

- (b) Show that $F(x,y) = 2x+4y+1$ is a *primitive recursive function* on $\mathbb{N} \times \mathbb{N}$. by finding primitive recursive functions g and h such that $F = \text{prec}(g,h)$.

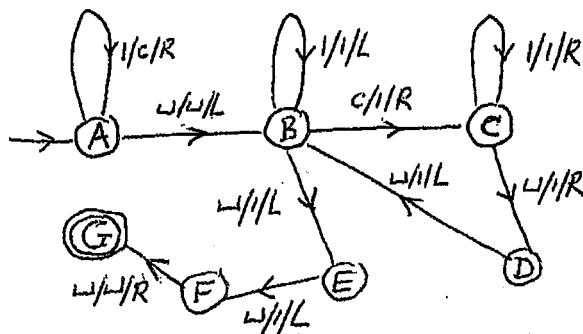
- (16) 4.(a) Define what "f is obtained by the *minimization* of the function $g: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ " means, and define what is a μ -*recursive partial function* on \mathbb{N}^n .

- (b) Let $f(x) = \text{Ceiling function of } [(2x+1)^{1/3}]$. Show that f is a μ -*recursive function*. [You may use the fact that PRED, MONUS, ADD, MULT & SIGN are primitive recursive if needed in #4, but you are *not allowed* to do so in Question #3.]

- (18) 5.(a) Define what is a *Turing-decidable binary relation*, R , on \mathbb{N} .

- (b) Show what happens at each step if (i) 1 and (ii) λ are the inputs for the TM M , shown on the right.

- (c) What is the *function computed* by M in monadic (base 1) notation?

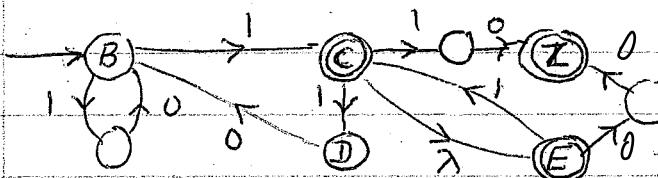


- (18) 6. Determine which of the following languages are regular and which are not.

$$(a) L_1 = \{a^k b^n : k \pmod{3} > (4 - n^2) \pmod{3}\} \quad (b) L_2 = \{b^k c^n : k > 4 + n^2\}.$$

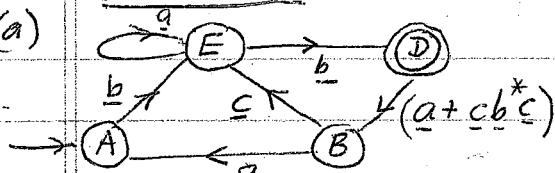
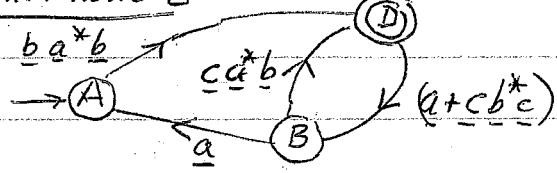
[If you say that it is regular, you must find a regular expression for it; if you say it is non-regular, you must give a complete proof.]

1(a)



(b) $G: \rightarrow A, A \rightarrow bE, E \rightarrow aE$
 $E \rightarrow bD, D \rightarrow aB, D \rightarrow cC, D \rightarrow \lambda$
 $C \rightarrow bC, C \rightarrow cB, B \rightarrow aA, B \rightarrow cE,$

2(a)

Eliminate C:Eliminate B: $R_1 = \emptyset$ Eliminate E

$$R_3 = (a+cb^*)a \quad (a+cb^*)(ca^*b) = R_4$$

$$\begin{aligned} \text{So } L(N) &= R_1 R_2 (R_4 + R_3 R_1 R_2)^* \\ &= \emptyset^* \cdot ba^*b \cdot [(a+cb^*) (ca^*b) + (a+cb^*)a \cdot \emptyset^* \cdot (ba^*b)]^* \end{aligned}$$

2(b) Let H_n = set of all TMs with n states & tape alphabet $\{1, \sqcup\}$ which halts when started on the blank tape. The busy beaver function is defined by $\beta(n)$ = maximum no. of 1's a TM on H_n can produce when started on the blank tape.

3(a) The initial functions are: the constant 0, the zero function $z(x) \equiv 0$, the successor function $s(x) = x+1$, and the projective functions $I_{k,n}(x_1, \dots, x_n) = x_k$ for $1 \leq k \leq n$ & $I_{0,n}(x_1, \dots, x_n) = \lambda$.

$F = \text{prec}(g, h)$ is the function $F: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ that is defined by putting $F(\underline{x}, 0) = g(\underline{x})$ & $F(\underline{x}, s(y)) = h(\underline{x}, y, F(\underline{x}, y))$. Here $\underline{x} = \langle x_1, \dots, x_n \rangle$

$$(b) F(\underline{x}, y) = 2x + 4y + 1. \text{ So } F(\underline{x}, 0) = 2x + 4(0) + 1 = 2x + 1. [\leftarrow g(\underline{x})]$$

$$\text{Also } F(\underline{x}, s(y)) = 2x + 4(y+1) + 1 = (2x + 4y + 1) + 4 = F(\underline{x}, y) + 4. [\leftarrow h(\underline{x}, y, F(\underline{x}, y))]$$

$\therefore g(\underline{x}) = 2x + 1$. g is not an initial function, so we will write $g = \text{prec}(g_1, h_1)$

$$\text{& } h(\underline{x}, y, F(\underline{x}, y)) = F(\underline{x}, y) + 4. \therefore h = \underbrace{s_0 s_0 s_0 s_0}_{3rd} I_{3,3}$$

$$g(0) = 2(0) + 1 = 1 [\leftarrow g_1] \quad \& \quad g(y+1) = 2(y+1) + 1 = (2y+1) + 2 = g(y) + 2 [\leftarrow h_1(y, g(y))]$$

$$\therefore g = \text{prec}(s_0 0, s_0 s_0 I_{2,2}).$$

$$\text{Hence } F = \text{prec}(g, h) = \text{prec}(\text{prec}(s_0 0, s_0 s_0 I_{2,2}), s_0 s_0 s_0 I_{3,3}).$$

$\therefore F$ is a primitive recursive function.

4(a) $f: \mathbb{N}^n \rightarrow \mathbb{N}$ is obtained by the minimization of $g: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ means

$$f(x) = \begin{cases} \text{the smallest value of } y \text{ such that } g(x, y) = 0 \\ \text{undefined if } g(x, y) > 0 \text{ for each } y \in \mathbb{N}. \end{cases}$$

A μ -recursive partial function $f: \mathbb{N}^n \rightarrow \mathbb{N}$ is any partial function f that can be obtained from the initial functions by a finite no. of applications of compositions, cartesian products, primitive recursions and minimizations on total functions.

(b) Let $g(x, y) = (2x+1) \dot{-} y^3$. Then $F = \mu[g, 0]$

$$= \mu[\text{MONUS} \circ \{ s \circ \text{ADD} \circ (I_{1,2} \wedge I_{1,2}) \wedge \text{MULT} \circ (\text{MULT} \circ (I_{2,2} \wedge I_{2,2}), I_{2,2}) \}, 0].$$

5(a) A binary relation $R \subseteq \mathbb{N} \times \mathbb{N}$ is Turing decidable if we can find a TM M such that M halts in an accepting state if $(x, y) \in R$ and M halts in a non-accept state, $f(x, y) \notin R$ for all $(x, y) \in \mathbb{N} \times \mathbb{N}$.

$$(b) (i) \langle A, 1 \rangle \vdash \langle A, c \sqcup \rangle \vdash \langle B, c \rangle \vdash \langle C, 1 \sqcup \rangle \vdash \langle D, 11 \sqcup \rangle \vdash \langle B, 111 \rangle \\ \vdash \langle B, 111 \rangle \vdash \langle B, \sqcup 111 \rangle \vdash \langle E, \sqcup 1111 \rangle \vdash \langle F, \sqcup 11111 \rangle \vdash \langle G, 11111 \rangle$$

$$(ii) \langle A, \sqcup \rangle \vdash \langle B, \sqcup \sqcup \rangle \vdash \langle E, \sqcup 1 \sqcup \rangle \vdash \langle F, \sqcup 11 \rangle \vdash \langle G, 11 \rangle.$$

(c) $f(0) = 2$, $f(1) = 5$, verify $f(2) = 8$. So $f(n) = 3n + 2$.

6(a) REGULAR. If $n \equiv 0 \pmod{3}$, $4-n^2 \equiv 4-0 \equiv 1 \pmod{3}$. So $k \equiv 2 \pmod{3}$

If $n \equiv 1 \pmod{3}$, then $4-n^2 \equiv 4-1^2 \equiv 0 \pmod{3}$. So $k \equiv 1 \text{ or } 2 \pmod{3}$.

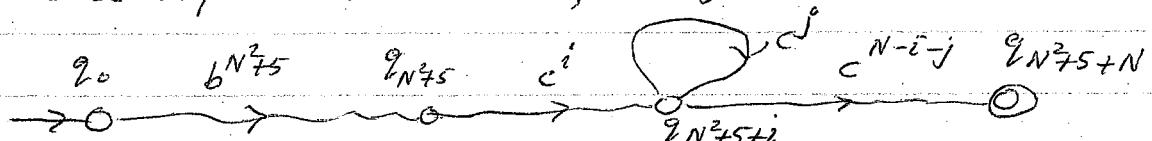
If $n \equiv 2 \pmod{3}$, then $4-n^2 \equiv 4-2^2 \equiv 0 \pmod{3}$. So $k \equiv 1 \text{ or } 2 \pmod{3}$

i. $\alpha(aaa)^*(bbb)^* + (a+a)(aaa)^*(bbb)^* \not\in \text{L}_1$ + $(a+a)(aaa)^*(bbb)^* b^j \not\in \text{L}_1$ a reg. expr. for L_1

(b) NON-REGULAR. Suppose L_2 was regular. Then we can find a

λ -free NFA M with N states such that $\mathcal{L}(M) = L_2$. Now $b^{N^2+5} c^N \in L_2$,

because $N^2+5 > 4+(N)^2$, so it will be accepted by M . Since it takes $N+1$ states to process c^N , the acceptance track of $b^{N^2+5} c^N$ must have a loop as shown below, with $j \geq 1$



Now if we ride this loop twice, we will see that M accepts the string $b^{N^2+5} c^i c^j c^{N-i-j} = b^{N^2+5} c^{N+j}$. But $N^2+5 \neq 4+(N+j)^2$ because $4+(N+j)^2 = 4+N^2+2Nj+j^2 \geq N^2+4+2+1$ since $j \geq 1$.

Hence $b^{N^2+5} c^{N+j} \notin L_2$. But this contradicts the fact that $b^{N^2+5} c^{N+j} \in \mathcal{L}(M) = L_2$. Hence L_2 must be non-regular. END