

Answer all 8 questions. An unjustified answer will receive little or no credit. No calculators, formula sheets, or cell-phones are allowed. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(12) 1. Let $f(x) = 2x/(x-1)$ and $g(x) = x/(3x-1)$.

- Find $(f \circ g)(x)$ and simplify your answer as far as possible.
- Find $\text{dom}(f \circ g)$.

(12) 2. Let $f(x) = (2x-3)/(x+1)$.

- Find $f^{-1}(x)$ and $\text{dom}(f^{-1})$
- Check that $f^{-1}(f(x)) = x$.

(12) 3(a) Define what " $\lim_{x \rightarrow a} f(x) = L$ " means in terms of ϵ and δ .

(b) Let $f(x) = 6x+1$, $L = 4$, and $a = 1/2$. Prove that $\lim_{x \rightarrow 1/2} (6x+1) = 4$ by using the $\epsilon-\delta$ method.

(12) 4(a) Let f be a function. Define what is the derived function $f'(x)$ and what is $\text{dom}(f')$.

(b) Let $f(x) = 1/\sqrt{x}$. Find $f'(x)$ directly from the definition of $f'(x)$ that you gave in part (a).

(12) 5 (a) The position of a particle moving on the x -axis is given by $x(t) = t^2 + t$. Find the average velocity between times $t = 1.0$ & $t = 1.1$. Also find the instantaneous velocity at time $t = 1$.

(b) Find $\lim_{x \rightarrow 4} \left\{ \sin^{-1} \left(\frac{2\sqrt{x} - 4}{x - 4} \right) \right\}$

(12) 6. Evaluate the following derivatives and simplify your answers as far as possible.

(a) $\frac{d}{dx} \left\{ (x^2 + 1) \cdot (x^2 + 3) \right\}$

(b) $\frac{d}{dx} \left(\frac{x^2 - 1}{x^2 + 1} \right)$

(14) 7(a) Find $\lim_{x \rightarrow +\infty} \sqrt{\left\{ \frac{4 + \sqrt{x}}{9 + 2x} - \frac{x^2}{4x^2} \right\}}$

(b) Find $\lim_{x \rightarrow +\infty} \left\{ \frac{1}{x - \sqrt{x^2 - 2x}} \right\}$

(14) 8 (a) Find $\lim_{x \rightarrow +\infty} \left(1 - \frac{1}{3x} \right)^{-3x/2}$

(b) Find $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x \cdot \sin x \cdot \cos x}$.

[You may use the fact that $\lim_{x \rightarrow +\infty} \left(1 + \frac{a}{x} \right)^x = e^a$
and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, if needed, in any problem.]

$$\begin{aligned} 1(a) \quad (f \circ g)(x) &= f(g(x)) = \frac{2 \cdot g(x)}{g(x)-1} = \frac{2 \cdot \{x/(3x-1)\}}{\{x/(3x-1)\}-1} \\ &= \frac{(2x)/(3x-1)}{\{x-(3x-1)\}/(3x-1)} = \frac{2x}{3x-1} \cdot \frac{3x-1}{-2x+1} = \frac{2x}{1-2x} \end{aligned}$$

$$\begin{aligned} 1(b) \quad \text{Dom}(f \circ g) &= \text{dom}(g) - \text{all } x \text{ for which } g(x) \text{ is not in dom}(f) \\ &= (\mathbb{R} - \{1/3\}) - \text{all } x \text{ for which } x/(3x-1) = 1 \\ &= \mathbb{R} - \{1/3, 1/2\}, \end{aligned}$$

because $\frac{x}{3x-1} = 1 \Rightarrow x = 3x-1 \Rightarrow 1 = 2x \Rightarrow x = 1/2$.

$$\begin{aligned} 2(a) \quad \text{Let } y = f(x). \quad \text{Then } f^{-1}(y) &= x. \quad \text{But } y = (2x-3)/(x+1) \text{ also,} \\ \text{So } y(x+1) &= 2x-3 \Rightarrow yx+y = 2x-3 \Rightarrow y+3 = 2x-yx \\ \Rightarrow (y+3) &= x(2-y) \Rightarrow x = (y+3)/(2-y). \\ \therefore f^{-1}(y) &= (y+3)/(2-y). \quad \therefore f^{-1}(x) = \frac{(x+3)/(2-x)}. \\ \text{dom}(f^{-1}) &= \mathbb{R} - \{2\}. \end{aligned}$$

$$\begin{aligned} 2(b) \quad f^{-1}(f(x)) &= \frac{f(x)+3}{2-f(x)} = \frac{(2x-3)/(x+1)+3}{2-(2x-3)/(x+1)} \\ &= \frac{\{(2x-3)+3(x+1)\}/(x+1)}{\{2(x+1)-(2x-3)\}/(x+1)} = \frac{5x}{x+1} \cdot \frac{x+1}{5} = x. \end{aligned}$$

3(a) $\lim_{x \rightarrow a} f(x) = L$ if for any $\varepsilon > 0$ we can find a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta$.

(b) Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/6$. Then whenever $0 < |x-1/2| < \delta$ we have

$$\begin{aligned} |f(x) - L| &= |6x+1-4| = |6x-3| = |6(x-1/2)| \\ &= 6|x-1/2| < 6 \cdot \delta = 6 \cdot (\varepsilon/6) = \varepsilon \end{aligned}$$

$\therefore |f(x) - L| < \varepsilon$ whenever $0 < |x-1/2| < \delta$. $\therefore \lim_{x \rightarrow 1/2} (6x+1) = 4$.

$$4(a) \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{and}$$

$\text{dom}(f') = \text{set of } x \text{ in } \text{dom}(f) \text{ for which this limit exists.}$

$$\begin{aligned}
 (b) \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h} \cdot \sqrt{x}} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h} \cdot \sqrt{x}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{(x - x-h)}{\sqrt{x+h} \cdot \sqrt{x}} \cdot \frac{1}{\sqrt{x} + \sqrt{x+h}} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{-h}{\sqrt{x+h} \cdot \sqrt{x}} \cdot \frac{1}{\sqrt{x} + \sqrt{x+h}} = \lim_{h \rightarrow 0} \frac{-1}{(\sqrt{x+h} \cdot \sqrt{x})(\sqrt{x} + \sqrt{x+h})} \\
 &= -1 / \left\{ \lim_{h \rightarrow 0} (\sqrt{x+h}, \sqrt{x}) \cdot \lim_{h \rightarrow 0} (\sqrt{x} + \sqrt{x+h}) \right\} \\
 &= -1 / \{(\sqrt{x+0}, \sqrt{x}) \cdot (\sqrt{x} + \sqrt{x+0})\} = \frac{-1}{x \cdot 2\sqrt{x}} = \frac{-1}{2x\sqrt{x}}
 \end{aligned}$$

5(a) Average velocity = (net displacement)/(time taken)

$$\begin{aligned}
 &= \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\{(1.1)^2 + (1.1)\} - \{(1.0)^2 + (1.0)\}}{1.1 - 1.0} \\
 &= \frac{1.21 + 1.1 - 2.0}{0.1} = \frac{2.31 - 2.0}{0.1} = \frac{0.31}{0.1} = 3.1
 \end{aligned}$$

$$\begin{aligned}
 \text{Instantaneous velocity} &= x'(t) \Big|_{t=1} = (t^2 + t)' \Big|_{t=1} \\
 &= (2t+1) \Big|_{t=1} = 2(1) + 1 = 3.
 \end{aligned}$$

$$(b) \quad \lim_{x \rightarrow 4} \left\{ \sin^{-1} \left(\frac{2\sqrt{x} - 4}{x - 4} \right) \right\} = \sin^{-1} \left\{ \lim_{x \rightarrow 4} \frac{2(\sqrt{x} - 2)}{(x-4)} \right\} \text{ because } \sin^{-1}(x) \text{ is continuous}$$

$$\begin{aligned}
 &= \sin^{-1} \left\{ \lim_{x \rightarrow 4} \frac{2 \cdot (\sqrt{x} - 2)}{(x-4)} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \sin^{-1} \left\{ \lim_{x \rightarrow 4} \frac{2 \cdot (x-4)}{(x-4)(\sqrt{x}+2)} \right\} = \sin^{-1} \left\{ \lim_{x \rightarrow 4} \left(\frac{2}{\sqrt{x}+2} \right) \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \sin^{-1} \{ 2 / (\sqrt{4} + 2) \} = \sin^{-1}(1/2) = \pi/6.
 \end{aligned}$$

$$\begin{aligned}
 6(a) \frac{d}{dx} \{(x^2+1). (x^2+3)\} &= (x^2+1)' . (x^2+3) + (x^2+1) . (x^2+3)' \\
 &= (2x+0)(x^2+3) + (x^2+1)(2x+0) = 2x(x^2+3+x^2+1) \\
 &= 2x(2x^2+4) = 4x(x^2+2).
 \end{aligned}$$

$$\begin{aligned}
 6(b) \frac{d}{dx} \left(\frac{x^2-1}{x^2+1} \right) &= \{(x^2-1)' . (x^2+1) - (x^2-1) . (x^2+1)'\} / (x^2+1)^2 \\
 &= \{2x.(x^2+1) - 2x(x^2-1)\} / (x^2+1)^2 = 4x / (x^2+1)^2
 \end{aligned}$$

$$\begin{aligned}
 7(a) \lim_{x \rightarrow \infty} \sqrt{\frac{4+\sqrt{x}-x^2}{9+2x-4x^2}} &= \sqrt{\lim_{x \rightarrow \infty} \frac{4+x^{1/2}-x^2}{9+2x-4x^2}} \\
 &= \sqrt{\lim_{x \rightarrow \infty} \frac{4x^{-2}+x^{-3/2}-1}{9x^{-2}+2x^{-1}-4}} = \sqrt{\frac{0+0-1}{0+0-4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}.
 \end{aligned}$$

$$\begin{aligned}
 7(b) \lim_{x \rightarrow \infty} \frac{1}{x-\sqrt{x^2-2x}} &= \lim_{x \rightarrow \infty} \frac{1}{x-\sqrt{x^2-2x}} \cdot \frac{x+\sqrt{x^2-2x}}{x+\sqrt{x^2-2x}} \\
 &= \lim_{x \rightarrow \infty} \frac{x+\sqrt{x^2-2x}}{x^2-(x^2-2x)} = \lim_{x \rightarrow \infty} \frac{x+\sqrt{x^2-2x}}{2x} \\
 &= \lim_{x \rightarrow \infty} \frac{x+x\sqrt{1-2x^{-1}}}{2x} = \lim_{x \rightarrow \infty} \frac{1+\sqrt{1-2x^{-1}}}{2} = \frac{1+1}{2} = 1
 \end{aligned}$$

$$\begin{aligned}
 8(a) \lim_{x \rightarrow \infty} \left(1 - \frac{1}{3x}\right)^{-3x/2} &= \lim_{x \rightarrow \infty} \left\{ \left(1 + \frac{-1/3}{x}\right)^x \right\}^{-3/2} \\
 &= \left\{ \lim_{x \rightarrow \infty} \left(1 + \frac{-1/3}{x}\right)^x \right\}^{-3/2} = (e^{-1/3})^{-3/2} = e^{1/2} = \sqrt{e}
 \end{aligned}$$

$$\begin{aligned}
 8(b) \lim_{x \rightarrow 0} \frac{1-\cos x}{x \cdot \sin x \cdot \cos x} &= \lim_{x \rightarrow 0} \frac{1-\cos x}{x \cdot \sin x \cdot \cos x} \cdot \frac{1+\cos x}{1+\cos x} \\
 &= \lim_{x \rightarrow 0} \frac{1-\cos^2 x}{x \cdot \sin x \cdot \cos x \cdot (1+\cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x \cdot \sin x \cdot \cos x \cdot (1+\cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\sin x} \cdot \frac{1}{\cos x (1+\cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot 1 \cdot \lim_{x \rightarrow 0} \frac{1}{\cos x (1+\cos x)} \\
 &= 1 \cdot 1 \cdot \frac{1}{\cos(0)(1+\cos(0))} = \frac{1}{1(1+1)} = \frac{1}{2}.
 \end{aligned}$$