

Answer all 6 questions. No Calculators, class-notes, or formula-sheets are allowed. An unjustified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.

- (15) 1.(a) How many integers in  $\{1, 2, 3, \dots, 1000\}$  are divisible by *neither* 12 nor 20?  
 (b) Find the permutation which has  $\langle 2, 1, 3, 0, 1, 0 \rangle$  as its inversion sequence and check that your answer is correct.
- (16) 2.(a) Write down what is the *Multinomial Theorem* in full details. Then use it to find the coefficient of  $x^6y^4z^2$  in the expansion of  $(3x^2y - 5y + z/2)^6$ .  
 (b) Find the number of integer solutions of the equation:  $x_1 + x_2 + x_3 + x_4 = 11$  with  $x_1 \geq 4$ ,  $x_2 \geq -3$ ,  $x_3 \geq 5$ , and  $x_4 \geq -1$ .
- (16) 3.(a) Write down *both* versions of the *Inclusion-Exclusion Theorem*.  
 (b) Find the number of 20-combinations of the multi-set  $F = [17.a, 6.b, 9.c]$ .  
 [You should leave your answer in terms of simplified Binomial coefficients.]
- (18) 4. (a) How many permutations of  $\{2, 3, 4, 5, 6, 7, 8, 9\}$  have *each even number* going to an *even number* and *no odd number* going to itself?  
 (b) How many permutations of  $\{1, 3, 4, 5, 6, 7, 8, 9\}$  have *no even number* going to itself? [You may leave your answer in terms of simplified factorials.]
- (15) 5.(a) Define what is a *positive set* and what is an *ultimate set* with respect to the subsets  $A_1, A_2, \dots, A_n$  of  $U$ .  
 (b) Let  $M = [n_1.a_1, n_2.a_2, \dots, n_k.a_k]$  be a finite multi-set and put  $n = n_1 + n_2 + \dots + n_k$ . Prove that number of  $n$ -permutations of  $M$  = number of  $(n-1)$ -permutations of  $M$ .
- (20) 6.(a) Prove that for  $0 \leq k \leq n$ ,  $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \dots + \binom{n+k}{k} = \binom{n+k+1}{k}$ .  
 (b) Let  $n$  be a positive integer. Give a *combinatorial proof* that  

$$\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \dots + \binom{2n}{2n} = \binom{2n}{1} + \binom{2n}{3} + \binom{2n}{5} + \dots + \binom{2n}{2n-1}$$
  
 [You may use Pascal's formula without proof, if needed, in any problem.]

Solutions to Test #1

Fall 2020

1(a) Let  $U = \{1, 2, 3, \dots, 1000\}$ ,  $A = \{x \in U : x \text{ is divisible by } n\}$ . Then our answer is  $|A_{12}^c \cap A_{20}^c| = |U| - |A_{12}| - |A_{20}| + |A_{12} \cap A_{20}|$  bec.  $A_{60} = A_{12} \cap A_{20}$   
 $= 1000 - \lfloor \frac{1000}{12} \rfloor - \lfloor \frac{1000}{20} \rfloor + \lfloor \frac{1000}{60} \rfloor = 1000 - 83 - 50 + 16 = 883$

(b) Write down  $\langle 6 \rangle$  because the inversion seq. has length 6  
 Then  $\langle 6, 5 \rangle$  bec.  $i_5 = 1$  then  $\langle 4, 2, 6, 5, 3 \rangle$  bec.  $i_2 = 1$   
 Then  $\langle 4, 6, 5 \rangle$  bec.  $i_4 = 0$  then  $\langle 4, 2, 1, 6, 5, 3 \rangle$  bec.  $i_1 = 2$   
 then  $\langle 4, 6, 5, 3 \rangle$  bec.  $i_3 = 3$  Check: inv. seq. =  $\langle 2, 1, 3, 0, 1, 0 \rangle$

2(a) For any  $k, n \in \mathbb{N}$ ,  $(x_1 + \dots + x_k)^n = \sum_{\langle n_1, \dots, n_k \rangle : n_1 + \dots + n_k = n} \binom{n}{i_1, i_2, \dots, i_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$   
 where  $\binom{n}{n_1, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$

The coeff. of  $x^6 y^4 z^2$  in the expansion of  $[3x^2y + (-5y) + (z/2)]^6$   
 $=$  Coeff. of term formed from  $(3x^2y)^3 \cdot (-5y)^1 \cdot (z/2)^2$   
 $= \binom{6}{3, 1, 2} \cdot (3)^3 \cdot (-5)^1 \cdot \left(\frac{1}{2}\right)^2 = \frac{-6 \cdot 5 \cdot 4 \cdot 3! \cdot 3^3 \cdot 5}{3! 1! 2! \cdot 4} = -2,025$

(b) Put  $y_1 = x_1 - 4$ ,  $y_2 = x_2 + 3$ ,  $y_3 = x_3 - 5$ ,  $y_4 = x_4 + 1$ . Then the eq.  $x_1 + x_2 + x_3 + x_4 = 11$  becomes  $(y_1 + 4) + (y_2 - 3) + (y_3 + 5) + (y_4 - 1) = 11$ .  
 So  $y_1 + y_2 + y_3 + y_4 = 11 - 5 = 6$ , and  $y_i \geq 0$  for each  $i = 1, 2, 3, 4$ .  
 So no. of integer solutions of  $x_1 + x_2 + x_3 + x_4 = 11$  will  
 no. of integer solutions of  $y_1 + y_2 + y_3 + y_4 = 6$  with  $y_i \geq 0$   
 $= \binom{6+4-1}{4-1} = \binom{9}{3} = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{3! 6!} = 84$

3(a) Let  $A_1, A_2, \dots, A_n$  be subsets of  $U$ . Then

(Version 1)  $|A_1^c \cap A_2^c \cap \dots \cap A_n^c| = \sum_{k=0}^n (-1)^k \sum_{\langle i_1, \dots, i_k \rangle : 1 \leq i_1 < i_2 < \dots < i_k \leq n \text{ (incl. empty seq.)}} |U \cap A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|$

(Version 2)  $|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \sum_{\langle i_1, \dots, i_k \rangle : 1 \leq i_1 < i_2 < \dots < i_k \leq n \text{ excl. empty seq.}} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|$

3(b) Let  $U$  = set of all 20-comb. of  $[\infty, a, \infty, b, \infty, c] = M$ , and  
 $A$  = set of all 20-comb. in  $U$  with  $\geq 18$ 's. Then  $|A| = |\text{set of 2-comb. of } M|$

$B$  = set of all 20-comb. in  $U$  with  $\geq 7$ 's.  $|B| = |\text{set of 13-comb. of } M|$

$C$  = set of all 20-comb. in  $U$  with  $\geq 10$ 's.  $|C| = |\text{set of 10-comb. of } M|$

Also  $|B \cap C| = |\text{set of 3-comb. of } M|$ ,  $A \cap B = A \cap C = A \cap B \cap C = \emptyset$ .

$\therefore$  No. of 20-comb. of  $F = |A^c \cap B^c \cap C^c| = |U| - |A| - |B| - |C|$

$+ |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$

$$= \binom{20+3-1}{3-1} - \binom{2+3-1}{3-1} - \binom{13+3-1}{3-1} - \binom{10+3-1}{3-1} + \binom{3+3-1}{3-1} + 0 + 0 - 0$$

$$= \binom{22}{2} - \binom{4}{2} - \binom{15}{2} - \binom{12}{2} + \binom{5}{2}.$$

4(a) If the 4 even numbers go to the 4 even nos., then the odd numbers can only go to the odd nos. So answer = (no. of perm. of  $\{2, 4, 6, 8\}$ )  $\cdot$  (no. of derangements of  $\{3, 5, 7, 9\}$ )  
 $= 4! \cdot D_4 = (24)(9) = 216$ .

(b) Let  $A_n$  = no. of perm. with  $n$  going to itself. Then

$$\text{Answer} = |A_4^c \cap A_6^c \cap A_8^c| = |U| - |A_4| - |A_6| - |A_8| + |A_4 A_6| + |A_4 A_8| + |A_6 A_8| - |A_4 A_6 A_8|$$

where  $U$  = set of all permutations of  $\{1, 3, 4, 5, 6, 7, 8, 9\}$

Now  $|U| = 8!$ ,  $|A_4| = |A_6| = |A_8| = 7!$ ,  $|A_4 A_6| = |A_4 A_8| = |A_6 A_8| = 6!$

and  $|A_4 A_6 A_8| = 5!$ . So answer =  $8! - 3(7!) + 3(6!) + 5!$

5(a) A positive set is an expression of the form  $U \cap A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}$  where  $\langle i_1, i_2, \dots, i_k \rangle$  is a subsequence of  $\langle 1, 2, 3, \dots, n \rangle$  with the empty subsequence included.

An ultimate set is an expression of the form  $X_1 \cap X_2 \cap X_3 \cap \dots \cap X_n$  where  $X_i = A_i$  or  $A_i^c$ .

(b) Number of  $(n-1)$ -permutations of  $M$

$$= \text{no. of perm. with } (n_1-1) a_1 \text{'s} + \text{no. with } (n_2-1) a_2 \text{'s} + \dots + \text{no. of perm. with } (n_3-1) a_3 \text{'s} + \dots + \text{no. with } (n_k-1) a_k \text{'s}$$

$$\begin{aligned}
 5(b) &= \frac{(n-1)!}{(n_1-1)! n_2! \dots n_k!} + \frac{(n-1)!}{n_1! (n_2-1)! \dots n_k!} + \dots + \frac{(n-1)!}{n_1! n_2! \dots (n_k-1)!} \\
 &= (n-1)! \left[ \frac{n_1}{n_1! n_2! \dots n_k!} + \frac{n_2}{n_1! n_2! \dots n_k!} + \dots + \frac{n_k}{n_1! n_2! \dots n_k!} \right] \\
 &= \frac{(n_1+n_2+\dots+n_k)(n-1)!}{n_1! n_2! \dots n_k!} = \frac{n!}{n_1! n_2! \dots n_k!} = \text{No. of } n\text{-perm of } M.
 \end{aligned}$$

6(a) We shall prove the result by parametric induction on  $k$ .

[ $n$  will be the parameter.] For  $k=0$ , we get  $\binom{n}{0} = 1 = \binom{n+1}{0}$ .

So the result is true for  $n=0$ . Assume it is true for  $k$ .

Then  $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \dots + \binom{n+k}{k} = \binom{n+k+1}{k}$ . So

$$\binom{n}{0} + \binom{n+1}{1} + \dots + \binom{n+k}{k} + \binom{n+k+1}{k+1} = \binom{n+k+1}{k} + \binom{n+k+1}{k+1} = \binom{n+k+1+1}{k+1}$$

by Pascal's identity. So if the result is true for  $k$ , it will be true for  $k+1$ . Since  $n$  was arb., it follows by the Principle of Parametric Induction that it is true for all  $k, n$ .

(b) Let  $E =$  set of subsets of  $\{1, 2, 3, \dots, 2n\}$  with even no. of elements and  $D =$  set of subsets of  $\{1, 2, 3, \dots, 2n\}$  with odd no. of elements.

Define  $f: E \rightarrow D$  by  $f(S) = \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S. \end{cases}$

Also define  $g: D \rightarrow E$  the same way  $g(S) = S - \{1\}$

if  $1 \in S$  and  $g(S) = S \cup \{1\}$  if  $1 \notin S$ . Then

$f \circ g: D \rightarrow D =$  identity function &  $g \circ f: E \rightarrow E =$  identity function.

$\therefore g = f^{-1}$ . Hence  $f$  is a bijection. So

$$\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \dots + \binom{2n}{2n} = |E|$$

$$= |D| = \binom{2n}{1} + \binom{2n}{3} + \dots + \binom{2n}{2n-1}$$

subsets with 1 element \ 3 elements ... \ (2n-1) elements

$$\binom{2n}{0} + \binom{2n}{2} + \dots + \binom{2n}{2n}$$

subsets with 0 elements 2 elements 2n elements END