

Answer all 6 questions. No calculators or class-notes are allowed. An unjustified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.

- (15) 1. Find the solution of the recurrence equation $x_{n+1} - 2x_n = 3$ with the initial condition $x_0 = 1$.
- (15) 2. Find the complementary solution and give (with reasons) the minimum form of a particular solution of each of the following recurrence equations.
(a) $(E^2 + 9I)(E^2 + 3E - 4I)(x_n) = 7n^2$.
(b) $(E^2 - 4E + 4I)(E^2 - 4E + 5I)(x_n) = 3n.(2)^n$.
- (20) 3.(a) Use the method of generating functions to find the solution of the recurrence equation: $a_n - a_{n-1} - 2a_{n-2} = 0$ with $a_0 = 5$ and $a_1 = 4$.
(b) Let $M = [\infty.a, \infty.b, 11.c]$. Find the generating function of the collection of all n -combinations of M with an even number of a's, at least 5 b's, and an odd numbers of c's. [Simplify your answer as far as possible.]
- (15) 4.(a) Define what is the zero-column (diagonal) of a sequence $\{h_k\}_{k \in \mathbb{N}}$.
(b) Let $h_k = 3k^2 + k - 2$. Find the zero-column of $\{h_k\}_{k \in \mathbb{N}}$ and use it to get a formula for the sum $h_0 + h_1 + h_2 + \dots + h_n$. [Simplify your answer as far as possible.]
- (15) 5.(a) Let S be a subset of $\{1, 2, 3, \dots, 2n+2\}$ with $n+2$ elements. Prove that we can always find 2 integers in S such that one is an integer-multiple of the other.
(b) Define what are the Stirling coefficients of the First & Second kinds.
- (20) 6.(a) Let $s(p, k)$ be the number of circular permutations of $\{1, 2, 3, \dots, p\}$ with k cycles. Prove that $s(p+1, k) = s(p, k-1) + p \cdot s(p, k)$.
(b) Prove for any k with $1 \leq k \leq p$, the Stirling coefficients of the Second kind satisfy $\binom{p+1}{k} = \binom{p}{k-1} + k \cdot \binom{p}{k}$.

[In questions #5 and #6, you must prove everything by using the definitions. You are not allowed to use any similar-looking theorem proved in class.]

Solutions to Test #2

Fall 2020

1. The homog. eq. is $(E-2I)x_n = 0$. So $x_n^c = A \cdot (2)^n$

Since 1 is not a root of the auxiliary equation

try $x_n^P = b$. Then $x_{n+1}^P = b$. So $x_{n+1} - 2x_n = 3$

becomes $b - 2b = 3$. $\therefore b = -3$. So $x_n^c = -3$.

$\therefore x_n = x_n^c + x_n^P = A \cdot (2)^n - 3$. Now $x_0 = 1$. So

$$A \cdot (2)^0 - 3 = 1 \quad \therefore A = 1 + 3 = 4 \quad \therefore x_n = 4 \cdot (2)^n - 3$$

$$2(a) (E^2 + 9I)(E^2 + 3E - 4I) = 0 \Rightarrow (E-3i)(E+3i)(E+4)(E-1) = 0$$

$$\therefore E = 3i, -3i, -4 \text{ or } 1 \quad \therefore x_n^c = A \cdot (3i)^n + B \cdot (-3i)^n + C \cdot (-4)^n + D \cdot (1)^n$$

Since $7n^2 \cdot (1)^n$ is a polynomial of deg. 2 & 1 is a root of the aux. eq. of multiplicity 1, the minimal form of a particular solution will be $x_n^P = (a + bn + cn^2) \cdot n^1 \cdot (1)^n$.

$$(b) (E^2 - 4E + 4)(E^2 - 4E + 5I) = 0 \Rightarrow (E-2)^2 [(E-2)^2 + 1] = 0$$

$$\therefore E = 2(\text{twice}), 2+i, \text{ and } 2-i \quad \therefore x_n^c = (A+Bn)(2)^n + C \cdot (2+i)^n + D \cdot (2-i)^n$$

Since $3n \cdot (2)^n$ is a polynomial of deg. 1 & 2 is a root of the aux. eq. of mult. 2, the minimal form of a particular solution will be $x_n^P = (a + bn) \cdot n^2 \cdot (2)^n$.

$$3(a) \text{ Let } f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$$

$$\text{Then } -x \cdot f(x) = -a_0x - a_1x^2 - \dots - a_{n-1}x^n - \dots$$

$$-2x^2 f(x) = -2a_0x^2 - \dots - 2a_{n-2}x^n - \dots$$

$$\therefore (1-x-2x^2)f(x) = a_0 + (a_1-a_0)x + 0 \cdot x^2 + \dots + 0 \cdot x^n + \dots = 5-x$$

$$\therefore f(x) = \frac{5-x}{1-x-2x^2} - \frac{A}{(1+x)} + \frac{B}{1-2x} \Rightarrow 5-x = A(1-2x) + B(1+x)$$

$$\text{Putting } x=-1 \text{ gives } 5-(-1) = A(1-(-2)) + B(0) \Rightarrow A=2$$

$$\text{Putting } x=\frac{1}{2} \text{ gives } 5-\frac{1}{2} = A(0) + B\left(1+\frac{1}{2}\right) \Rightarrow B=-3$$

$$\therefore f(x) = \frac{2}{1+x} + \frac{-3}{1-2x} = 2[1+(-x)+(-x)^2+\dots+(-x)^n+\dots]$$

$$\therefore a_n = 2 \cdot (-1)^n + 3 \cdot (2)^n$$

3(b) The generating function of the no. of n -comb. of M

$$= (x + x^2 + x^4 + \dots)(x^5 + x^6 + x^7 + \dots). (x^1 + x^3 + x^5 + \dots + x^n)$$

$$= 1/(1-x^2) \cdot x^5 [1/(1-x)] \cdot x \cdot [1 - (x^2)^{5+1}] / (1-x^2)$$

$$= \frac{x^6}{(1-x)(1+x)} \cdot \frac{1}{(1-x)} \cdot \frac{1-x^{12}}{(1-x)(1+x)} = \frac{x^6 (1-x^{12})}{(1-x)^3 (1+x)^2}.$$

4(a) The zero column of $\langle h_k \rangle_{k \in N}$ is just $\langle \Delta^k h_0 \rangle_{k \in N}$.

$$(b) \quad h_k = -2 \quad 2 \quad 12 \quad 28 \quad 50 \quad 78$$

$$\Delta h_k = 4 \quad 10 \quad 16 \quad 22 \quad 28 \quad \dots$$

$$\Delta^2 h_k = 6 \quad 6 \quad 6 \quad 6 \quad \dots$$

$$\Delta^3 h_k = 0 \quad 0 \quad 0 \quad \dots$$

$$\therefore h_k = -2 \binom{k}{0} + 4 \binom{k}{1} + 6 \binom{k}{2}. \quad \therefore \sum_{k=0}^n h_k = -2(n+1) + 4 \binom{n+1}{2} + 6 \binom{n+1}{3}$$

$$= (n+1)[(-2) + 4n/2! + 6n(n-1)/3!] \quad .$$

$$= (n+1)[-2 + 2n + n^2 - n] = (n+1)(n-1)(n+2).$$

5(a) Let $f: S \rightarrow \mathbb{N}$ be defined $f(s) = \text{odd part of } s$.

Since $1, 3, 5, \dots, 2m+1$ are the $n+1$ choices for $f(s)$ and S has $n+2$ elements, we must have two elements $s_1, s_2 \in S$ with $f(s_1) = f(s_2)$. So $s_1 = 2^a \cdot c$ and $f(s_2) = 2^b \cdot c$ where c is odd. Now clearly the smaller of s_1, s_2 will divide the larger of s_1, s_2 .

[Note: Since $s_1 \neq s_2$, $a \neq b$. So $2^a \cdot c$ divides $2^b \cdot c$ if $a < b$; and $2^b \cdot c$ divides $2^a \cdot c$ if $a > b$.]

(b) The Stirling coefficients of the First kind are the unique integers $[n]_p$ such that $[n]_p = \sum_{k=0}^p (-1)^{p-k} \begin{Bmatrix} p \\ k \end{Bmatrix} \cdot n^k$

The Stirling coefficients of the Second kind are the unique integers $\{n\}_k$ such that $n^p = \sum_{k=0}^p \{n\}_k \cdot [n]_k$. Here $[n]_k = n(n-1)(n-2) \cdots (n-(k-1))$.

6(a) Let \mathcal{A} = set of seatings of $\{1, 2, 3, \dots, p+1\}$ at k indistinguishable tables with no table empty. Then $|\mathcal{A}| = s(p, k)$.
 Let \mathcal{B} = set of seatings in \mathcal{A} with $p+1$ by itself, and
 \mathcal{C} = set of all seatings in \mathcal{A} with $p+1$ not by itself.
 Then $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ & $\mathcal{B} \cap \mathcal{C} = \emptyset$, so $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}|$. Now if we remove the table & $p+1$ from a seating in \mathcal{B} , we will get a seating of $\{1, 2, \dots, p\}$ at $k-1$ non-empty tables. And if we add $p+1$ at a new table to a seating of $\{1, 2, \dots, p\}$ at k non-empty tables, we will get a seating in \mathcal{B} . So $|\mathcal{B}| = s(p, k-1)$. Also if we remove $p+1$ from its table in a seating of \mathcal{C} , we will get a seating of $\{1, 2, \dots, p\}$ at k tables. And if we seat p to the right of $1, 2, \dots, p$ (m turns), in a seating of $\{1, 2, \dots, p\}$ at k non-empty tables, we will get p seatings in \mathcal{C} . So $s(p+1, k) = |\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| = s(p, k-1) + p \cdot s(p, k)$.

$$(b). \text{ We know } n^p = \sum_{k=0}^p \{P\}_k \cdot [n]_k. \text{ So } n^{p+1} = \sum_{k=0}^{p+1} \{P+1\}_k \cdot [n]_k. (*)$$

$$\text{Thus } n^{p+1} - n^p \cdot n = \sum_{k=0}^p \{P\}_k \cdot [n]_k \cdot (n) = \sum_{k=0}^p \{P\}_k \cdot [n]_k \cdot [(n-k)+k]$$

$$= \sum_{k=0}^p \{P\}_k \cdot [n]_k \cdot (n-k) + \sum_{k=0}^p \{P\}_k \cdot [n]_k \cdot k$$

$$= \sum_{k=0}^p \{P\}_k \cdot [n]_{k+1} + \sum_{k=0}^p k \cdot \{P\}_k \cdot [n]_k$$

$$\xrightarrow{k=1} = \sum_{k=1}^{p+1} \{P\}_{k-1} \cdot [n]_k + \sum_{k=1}^p k \cdot \{P\}_k \cdot [n]_k$$

$$\text{, replace } k \text{ by } k-1 \quad \text{delete } 0 \cdot \{P\}_0 \cdot [n]_0.$$

$$= \{P\}_{p+1} + \sum_{k=1}^p \left[\{P\}_{k-1} \cdot [n]_k + k \cdot \{P\}_k \cdot [n]_k \right] \quad (**)$$

Comparing coefficients in $(*)$ & $(**)$ for $1 \leq k \leq p$, we see that $\{P+1\}_k = \{P\}_{k-1} + k \cdot \{P\}_k$ & $\{P+1\}_p = \{P\}_p$.