

*Answer all 6 questions. No Calculators, class-notes, or formula-sheets are allowed. An unjustified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.*

- (16) 1.(a) How many integers in  $\{1, 2, 3, \dots, 800\}$  are divisible by **neither** 15 nor 20?  
 (b) Find the *permutation* which has  $\langle 4, 2, 3, 1, 1, 0 \rangle$  as its *inversion sequence* and check that your answer is correct.
- (16) 2.(a) Write down what is the **Multinomial Theorem** in full details. Then use it to find the *coefficient of  $x^5y^2z^3$*  in the expansion of  $(4x^2 + y/3 - xz/2)^6$ .  
 (b) Find the number of **integer solutions** of the equation  $x_1 + x_2 + x_3 + x_4 = 10$  with  $x_1 \geq -4, x_2 \geq 3, x_3 \geq -1, \text{ and } x_4 \geq 5$ .
- (16) 3.(a) Write down the *first version* of the **Inclusion-Exclusion Theorem**.  
 (b) Find the number of 20-combinations of the multi-set  $F = [5.a, 8.b, 11.c]$ . [You should leave your answer in terms of simplified Binomial coefficients.]
- (16) 4.(a) How many permutations of  $\{3, 4, 5, 6, 7, 8, 9\}$  have each *prime number* going to a *prime number and no non-prime number* going to itself?  
 (b) How many permutations of  $\{3, 4, 5, 6, 7, 8, 9\}$  have *no prime number* going to itself? [You should leave your answer in terms of simplified factorials.]
- (16) 5.(a) Let  $n$  be any *odd positive integer*. Give an **analytic proof** that  

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n}$$
.  
 (b) Give a **combinatorial proof** of the same result that is in part (a).
- (20) 6.(a) Define what is a *positive set* and what is an *ultimate set* with respect to the subsets  $A_1, A_2, \dots, A_n$  of the universal set  $U$ .  
 (b) Using the **Binomial Theorem**, find the value of the following two sums:  
 (i)  $1 \cdot \binom{n}{0} + 2 \cdot \binom{n}{1} + 3 \cdot \binom{n}{2} + \dots + (n+1) \cdot \binom{n}{n}$   
 (ii)  $\frac{1}{1} \cdot \binom{n}{0} + \frac{1}{2} \cdot \binom{n}{1} + \frac{1}{3} \cdot \binom{n}{2} + \dots + \frac{1}{n+1} \cdot \binom{n}{n}$

END

## Solutions to Test #1

Fall 2021

1 (a) Let  $U = \{1, 2, 3, \dots, 800\}$  &  $A_n = \{x \in U : x \text{ is divisible by } n\}$ . Then

$$A_{15} \cap A_{20} = A_{60} \text{ & our answer is } |A_{15}^c \cap A_{20}^c| = |U| - |A_{12}| - |A_{20}| + |A_{60}| \\ = 800 - \lfloor \frac{800}{12} \rfloor - \lfloor \frac{800}{20} \rfloor + \lfloor \frac{800}{60} \rfloor = 800 - 53 - 40 + 13 = \boxed{720}.$$

(b) Write down  $\langle 6 \rangle$  because the inversion seg. has length 6

Then  $\langle 6, 5 \rangle$  b/c.  $i_5 = 1$ ; then  $\langle 6, 4, 2, 5, 3 \rangle$  b/c.  $i_2 = 2$

then  $\langle 6, 4, 5 \rangle$  b/c.  $i_4 = 1$ ; and finally  $\langle 6, 4, 2, 5, 1, 3 \rangle$  b/c.  $i_1 = 4$ .

Then  $\langle 6, 4, 5, 3 \rangle$  b/c.  $i_3 = 3$ ; Check: inver. seg. =  $\langle 4, 2, 1, 6, 5, 3 \rangle$  ✓

$$2. (a) \text{For any } k, n \in \mathbb{N}; \quad (x_1 + \dots + x_k)^n = \sum_{(n_1, \dots, n_k)} \binom{n}{n_1, \dots, n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} \\ \text{where } \binom{n}{n_1, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}, \quad n_1 + n_2 + \dots + n_k = n$$

The coefficient of  $x^5 y^2 z^3$  in the expansion of  $[4x^2 + y/3 - xz/2]^6$

= coefficient of the term from  $(4x^2)^1 \cdot (y/3)^2 \cdot (-xz/2)^3$

$$= \binom{6}{1, 2, 3} \cdot 4 \cdot \left(\frac{1}{3}\right)^2 \cdot \left(-\frac{1}{2}\right)^3 = \frac{6 \cdot 5 \cdot 4 \cdot 1}{3 \cdot 2 \cdot 1} \cdot \frac{1}{9} \cdot \frac{(-1)}{8} = -\frac{30}{9} = \boxed{-\frac{10}{3}}$$

(b) Put  $x_1 = y_1 - 4$ ,  $x_2 = y_2 + 3$ ,  $x_3 = y_3 - 1$  &  $x_4 = y_4 + 5$ . Then the eq.

$x_1 + x_2 + x_3 + x_4 = 10$  becomes  $(y_1 - 4) + (y_2 + 3) + (y_3 - 1) + (y_4 + 5) = 10$ .

So  $y_1 + y_2 + y_3 + y_4 = 10 - 3 = 7$  with  $y_i \geq 0$  for each  $i = 1, 2, 3, 4$ .

So no. of integer solutions of  $x_1 + x_2 + x_3 + x_4 = 10$  with the various conditions on  $x_i$  will be no. of non-neg. sol. of  $y_1 + y_2 + y_3 + y_4 = 7$  and this is  $\binom{7+4-1}{4-1} = \binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = \frac{720}{6} = \boxed{120}$

3 (a) Let  $A_1, A_2, \dots, A_n$  be subsets of a universal set  $U$ . Then Version 1

$$\text{of the I.E.P is } |A_1^c \cap A_2^c \cap \dots \cap A_n^c| = \sum_{k=0}^n (-1)^k |\bigcup_{i=1}^n A_{i1} \cap A_{i2} \cap \dots \cap A_{ik}| \text{ where}$$

the 2nd summation is over all  $\langle i_1, \dots, i_k \rangle$   $1 \leq i_1 < i_2 < \dots < i_k \leq n$  (incl. empty seg.)

(b) Let  $M = [\infty, a, \infty, b, \infty, c]$  &  $U$  = set of all 20-comb. of  $M$ . Put

$A$  = set of all 20-comb. of  $M$  with  $\geq 6$  a's

= (set of all 14-comb. of  $M$ ) with 6a's added to each 14-comb,

$B$  = set of all 20-comb. of  $M$  with  $\geq 9$  b's

= (set of all 11-comb. of  $M$ ) with 11b's added to each 11-comb,

3(b) and  $C = \text{set of all } 20\text{-comb. of } M \text{ with } \geq 12 \text{ c's}$   
 $= (\text{set of all } 8\text{-comb. of } M) \text{ with } 12 \text{ c's added to each } 8\text{-comb.}$

Then  $A \cap B = \text{set of } 5\text{-comb. of } M \text{ with } [6.a, 9.b] \text{ added}$

$A \cap C = \text{set of } 2\text{-comb. of } M \text{ with } [6.a, 12.c] \text{ added}$

and  $B \cap C = \emptyset$ , and  $A \cap B \cap C = \emptyset$ . So our answer will  
be

$$\begin{aligned} |A^c \cap B^c \cap C^c| &= |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| \\ &= \binom{20+3-1}{3-1} - \binom{14+3-1}{3-1} - \binom{11+3-1}{3-1} - \binom{8+3-1}{3-1} + \binom{5+3-1}{3-1} + \binom{2+3-1}{3-1} \\ &= \binom{12}{2} - \binom{16}{2} - \binom{13}{2} - \binom{10}{2} + \binom{7}{2} + \binom{4}{2}. \end{aligned}$$

4(a) Let  $P = \{3, 5, 7\}$  &  $N = \{4, 6, 8, 9\}$ . Then  $P \cup N = U = \{3, 4, 5, 6, 7, 8, 9\}$

There are  $3!$  ways of permuting elements of  $P$  to  $P$

and  $D_4$  ways of deranging the elements of  $N$  into  $N$ .

So no. of perm. of  $U$  with primes going to primes and  
no non-prime going to itself  $= (3!) \cdot D_4 = 6(9) = \boxed{54}$

(b) For  $i = 3, 5$ , and  $7$ , let  $A_i = \text{set of permutations}$   
with  $i$  going to itself. Then  $|A_i| = 6!$  for  $i = 3, 5$  or  $7$

Also  $|A_i \cap A_j| = 5!$  for any two distinct  $i \& j \in \{3, 5, 7\}$

And  $|A_3 \cap A_5 \cap A_7| = 4!$  Finally  $|U| = 7!$  So number of  
perm. of  $U$  with no prime going to itself is

$$\begin{aligned} |A_3^c \cap A_5^c \cap A_7^c| &= |U| - |A_3| - |A_5| - |A_7| + |A_3 A_5| + |A_3 A_7| + |A_5 A_7| - |A_3 A_5 A_7| \\ &= 7! - 3(6!) + 3(5!) - 4!. \end{aligned}$$

5(a) We know  $n \in \mathbb{N}$  is odd and  $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

So putting  $x = -1$ , we get

$$0 = (1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n$$

Putting all the neg terms on LHS side we get

$$\binom{n}{1} - \binom{n}{3} + \dots - \binom{n}{n} = \binom{n}{0} + \binom{n}{2} + \dots + \binom{n}{n-1}$$

(b) Let  $E = \text{collection of all } k\text{-subsets of } \{1, 2, \dots, n\}$  with  $k$  even

&  $D = \dots \quad \dots \quad \dots$  with  $k$  odd.

5(b). Then define  $f: E \rightarrow D$  by  $f(S) = \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}$   
 and  $g: D \rightarrow E$  by  $g(S) = S - \{1\}$ , if  $1 \in S$  &  $g(S) = S \cup \{1\}$ , if  $1 \notin S$ .  
 Then  $fog: D \rightarrow D$  &  $gof: E \rightarrow E$  will be identity functions.

So  $g = f^{-1}$  and hence  $f: E \rightarrow D$  will be a bijection.

$\therefore |D| = |E|$ . But no we have

$$\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = |D| = |E| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1}.$$

6(a) A positive set w.r.t  $U$  &  $A_1, \dots, A_n$  is any expression of the form  $U^n A_{i_1} \cap \dots \cap A_{i_k}$  where  $\langle i_1, \dots, i_k \rangle$  is a subsequence of  $\langle 1, 2, \dots, n \rangle$

An ultimate set w.r.t.  $U$  &  $A_1, \dots, A_n$  is any expression of the form  $X_1 X_2 \dots X_n$  where  $X_i = A_i$  or  $A_i^c$ .

(b) (i) We know that  $\binom{n}{0} \cdot x^0 + \binom{n}{1} \cdot x^1 + \dots + \binom{n}{n} \cdot x^n = (1+x)^n$

$$\text{So } \binom{n}{0} \cdot x^0 + \binom{n}{1} \cdot x^1 + \dots + \binom{n}{n} \cdot x^{n+1} = x \cdot (1+x)^n$$

Differentiating both sides with respect to  $x$  gives us

$$1 \cdot \binom{n}{0} x^0 + 2 \cdot \binom{n}{1} x^1 + \dots + (n+1) \cdot \binom{n}{n} x^n = 1 \cdot (1+x)^n + nx \cdot (1+x)^{n-1}$$

Putting  $x=1$ , gives us

$$1 \cdot \binom{n}{0} + 2 \cdot \binom{n}{1} + 3 \cdot \binom{n}{2} + \dots + (n+1) \cdot \binom{n}{n} = 1 \cdot 2^n + n \cdot 1 \cdot 2^{n-1} = (n+2) \cdot 2^{n-1}$$

(ii) Again we know that  $\binom{n}{0} \cdot x^0 + \binom{n}{1} \cdot x^1 + \dots + \binom{n}{n} \cdot x^n = (1+x)^n$

So integrating both sides from 0 to 1 gives us

$$\int_0^1 \left\{ \binom{n}{0} \cdot x^0 + \binom{n}{1} \cdot x^1 + \dots + \binom{n}{n} \cdot x^n \right\} dx = \int_0^1 (1+x)^n dx$$

$$\therefore \left[ \binom{n}{0} \cdot \frac{x^1}{1} + \binom{n}{1} \cdot \frac{x^2}{2} + \dots + \binom{n}{n} \cdot \frac{x^{n+1}}{n+1} \right]_0^1 = \int_1^2 u^n du$$

Put  $u = (1+x)$ . Then  $du = dx$

and  $x=0 \Rightarrow u=1$ ,  $x=1 \Rightarrow u=2$

$$\text{So } \frac{1}{0!} \binom{n}{0} + \frac{1}{1!} \binom{n}{1} + \frac{1}{2!} \binom{n}{2} + \dots + \frac{1}{n+1!} \binom{n}{n} = \frac{2^{n+1} - 1}{n+1}.$$

END