

*Answer all 6 questions. No calculators or class-notes are allowed. An unjustified answer will receive little credit. Begin each of the 6 questions on 6 separate pages.*

- (16) 1. Find the *solution* of the recurrence equation  $x_{n+2} + 3x_{n+1} - 10x_n = 0$  with the *initial conditions*  $x_0 = 5$  and  $x_1 = -4$ .
- (18) 2. Find the *complementary solution* and give (with reasons) the *minimum form* of a *particular solution* of each of the following *recurrence equations*.
- (a)  $(E^2 + 4I)(E^2 + 2E - 3I)(x_n) = n^2 + 3n$ .  
(b)  $(E^2 - 2I)(E^2 - 2E + 5I)(x_n) = 5n.(2)^{n/2}$ .
- (18) 3.(a) Use the *method of generating functions* to find the solution of the recurrence equation  $a_n - 3a_{n-1} + 2 = 0$  with the initial condition  $a_0 = 3$ .  
(b) Let  $M = [\infty.a, 10.b, \infty.c]$ . Find the *generating function* of the collection of all *n-combinations* of  $M$  with an odd number of a's, an even number of b's, and at least 3 c's. [Simplify your answers as far as possible.]
- (16) 4.(a) Define what are the operators  $E$  and  $\Delta$  that operates on a sequence  $\langle h_k \rangle_{k \in \mathbb{N}}$ .  
(b) Let  $h_k = 3k^2 - 4k + 2$ . Find the *zero-column* of  $\langle h_k \rangle_{k \in \mathbb{N}}$  and use it to get a formula for the sum  $h_0 + h_1 + h_2 + \dots + h_n$ . [Simplify answer as far as possible.]
- (16) 5.(a) Write down what is the *function form* of the Pigeon-hole Principle.  
(b) Let  $n$  be a positive integer and  $\underline{s} = \langle a_1, a_2, a_3, \dots, a_{2n+1} \rangle$  be any sequence of  $2n+1$  integers. Prove that we can always find a *non-empty segment* (made up of consecutive terms) whose *terms* add up to a multiple of  $2n$ .
- (16) 6.(a) Define what are the *Stirling coefficients of the First & Second kinds*.  
(b) Prove for any  $k$  with  $1 \leq k \leq p-1$ , the Stirling coefficients of the *First kind* satisfy  $\begin{bmatrix} p \\ k \end{bmatrix} = \begin{bmatrix} p-1 \\ k-1 \end{bmatrix} + (p-1) \cdot \begin{bmatrix} p-1 \\ k \end{bmatrix}$ .

[In questions #5 and #6, you must prove everything by using the definitions. You are **not allowed** to use any similar-looking theorem proved in class.]

## Solutions to Test #2

Fall 2021

1. We have  $x_{n+2} + 3x_{n+1} - 10x_n = 0$ . So  $(E^2 - 3E + 10I)x_n = 0$ .

$$\therefore (E+5I)(E-2I)x_n = 0. \text{ Roots are } -5 \text{ & } 2. \text{ So } x_n = A(-5)^n + B(2)^n$$

$$\therefore x_0 = 5 = A+B \text{ and } x_1 = -4 = -5A + 2B. \text{ So } B = 5-A$$

and hence  $-4 = -5A + 2(5-A) \Rightarrow 3A = 6 \Rightarrow A = 2. \therefore B = 5-2 = 3$ .

Hence  $x_n = 2(-5)^n + 3(2)^n$ . Check:  $2(-5)^0 + 3(2)^0 = 5, 2(-5) + 3(2) = -4$

2(a)  $(E^2 - 4I)(E^2 + 2E - 3I) = 0 \Rightarrow (E-2i)(E+2i)(E+3)(E-1) = 0$ . So  $E = 2i, -2i, -3, \text{ or } 1$ .  $\therefore x_n^c = A.(2i)^n + B.(-2i)^n + C.(-3)^n + D.(1)^n$

Since  $n^2 + 3n$  is a polynomial of degree 2 & 1 is a root of the auxiliary equation of multiplicity 1, the minimal form of a particular solution will be  $x_n^p = (a + bn + cn^2).n!(1)^n$ .

(b)  $(E^2 - 2I)(E^2 - 2E + 5) = 0 \Rightarrow (E-\sqrt{2}i)(E+\sqrt{2}i)[(E-1)^2 + 4] = 0$ . So  $E = \sqrt{2}, -\sqrt{2}, 1+2i, \text{ or } 1-2i$ .  $\therefore x_n^c = A.(1+2i)^n + B.(1-2i)^n + C.(-\sqrt{2})^n + D.(\sqrt{2})^n$ .

Since  $5n(2)^{n/2} = 5n.(\sqrt{2})^n = (\text{a polynomial of deg. 1}).(\sqrt{2})^n$  &  $\sqrt{2}$  is a root of the aux. eq. of multiplicity 1, the minimal form of a particular solution will be  $x_n^p = (a + bn).n!(\sqrt{2})^n$ .

3(a) Let  $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$

$$\text{Then } -3x f'(x) = -3a_0x - 3a_1x^2 - \dots - 3a_{n-1}x^n - \dots$$

$$\text{and } 2/(1-x) = 2 + 2x + 2x^2 + \dots + 2x^n + \dots$$

$$\therefore (1-3x)f'(x) + 2/(1-x) = (a_0 + 2) + (a_1 - 3a_0 + 2)x + \dots + (a_n - 3a_{n-1} + 2)x^n + \dots \\ = (3+2) + 0x + 0x^2 + \dots + 0x^n + \dots = 5$$

$$\therefore (1-3x)f'(x) = 5 - \frac{2}{1-x} = (3-5x)/(1-x) \Rightarrow f'(x) = \frac{(3-5x)}{(1-3x)(1-x)} = \frac{1}{1-3x} + \frac{2}{1-x}$$

$$\therefore 3-5x = A(1-x) + B(1-3x). \text{ Putting } x=1 \text{ gives } -2 = B(-2) \Rightarrow B=1$$

$$\text{Putting } x = 1/3, \text{ gives } 3-5/3 = A(1-1/3) \Rightarrow 4/3 = 2A/3 \Rightarrow A=2$$

$$\therefore f(x) = \frac{2}{1-3x} + \frac{1}{1-x}. \text{ So } x_n = \text{coeff. of } x^n \text{ in the expansion}$$

$$2[1 + (3x) + (3x)^2 + \dots + (3x)^n + \dots] + 1.[1+x+x^2+\dots+x^n+\dots] = 2.(3)^n + 1$$

3(b) The generating function of the  $n$ -combinations of  $M$  (2)

$$\begin{aligned}
 &= (x^1 + x^3 + x^5 + \dots)(x^0 + x^2 + x^4 + \dots + x^{10})(x^3 + x^4 + x^5 + \dots) \\
 &= x^1 [1 + (x^2) + (x^2)^2 + \dots] \cdot [1 + (x^2) + (x^2)^2 + \dots + (x^2)^5] \cdot x^3 \cdot [1 + x + x^2 + \dots] \\
 &= \frac{x}{1-x^2} \cdot \frac{1-(x^2)^6}{1-x^2} \cdot \frac{x^3}{1-x} = \frac{x^4(1-x^{12})}{(1-x^2)^2(1-x)} = \frac{x^4(1-x^{12})}{(1-x)^3 \cdot (1+x)^2}
 \end{aligned}$$

4(a)  $E(\langle X_n \rangle_{n \in \mathbb{N}}) = \langle X_{n+1} \rangle_{n \in \mathbb{N}}$ ,  $\Delta(\langle X_n \rangle_{n \in \mathbb{N}}) = \langle X_{n+1} - X_n \rangle_{n \in \mathbb{N}}$

(b)

|                      | 0  | 1 | 2  | 3  | 4  | 5  |
|----------------------|----|---|----|----|----|----|
| $h_k = \Delta^0 h_k$ | 2  | 1 | 6  | 17 | 34 | 57 |
| $\Delta^1 h_k$       | -1 | 5 | 11 | 17 | 23 | .  |
| $\Delta^2 h_k$       | 6  | 6 | 6  | 6  | .  | .  |
| $\Delta^3 h_k$       | 0  | 0 | 0  | .  | .  | .  |

i. Zero column =  $\langle 2, -1, 6, 0, 0, 0, \dots \rangle$  since  $\deg[h(k)] = 2$

$$\begin{aligned}
 i. h_k &= 2 \binom{k}{0} - \binom{k}{1} + 6 \binom{k}{2} \quad \& \sum_{k=0}^n h_k = 2 \binom{n+1}{1} - \binom{n+1}{1} + 6 \binom{n+1}{2} \\
 &= 2(n+1) - 2(n+1)(n)/2 + 6(n+1)n(n-1) = (n+1) \left[ \frac{2n^2 - 3n + 1}{2} \right]
 \end{aligned}$$

5(a) Let  $f: P \rightarrow H$  be a function and suppose that  $P$  &  $H$  are finite sets with  $|P| > |H|$ . Then we can find  $x_1, x_2 \in P$  with  $x_1 \neq x_2$  such that  $f(x_1) = f(x_2)$ .

(b) Let  $P$  = set of all sums of the initial segments of  $S$

$$= \left\{ \sum_{k=1}^l a_k \mid \sum_{k=0}^{l-1} a_k = \{0, a_1, a_1+a_2, a_1+a_2+a_3, \dots, a_1+a_2+\dots+a_{2n+1}\} \right.$$

and  $H = \{0, 1, 2, 3, \dots, 2n-1\}$ . Define  $f: P \rightarrow H$  by  $f(x) = x \pmod{2n}$ .

So by the function form of the Pigeon-hole principle, we can find  $0 \leq i < j \leq 2n+1$  such that  $\sum_{k=1}^i a_k \equiv \sum_{k=1}^j a_k \pmod{2n}$

This is because  $|P| = 2n+2$  &  $|H| = 2n$ . with  $0 \leq i < j \leq 2n$

So we will get

$$x_1 + x_2 + x_3 + \dots + x_i \equiv x_1 + x_2 + x_3 + \dots + x_j \pmod{2n}$$

Hence  $x_{i+1} + x_{i+2} + \dots + x_j \equiv 0 \pmod{2n}$

i.  $\langle x_{i+1}, x_{i+2}, x_{i+3}, \dots, x_{j-1}, x_j \rangle$  will be a non-empty segment whose terms add up to a multiple of  $2n$ .

(3)

6(a) The Stirling coefficients of the first kind are the unique integers  $[P]_k$  such that  $[n]_p = \sum_{k=0}^p (-1)^{p-k} [P]_k \cdot n^k$

The Stirling coefficients of the second kind are the unique integers  $\{SP\}_k$  such that  $n^p = \sum_{k=0}^p \{SP\}_k [n]_k$ .  $[n]_k = n(n-1)(n-2)\dots(n-(k-1))$

(b) From the definition we know that

$$(*) [n]_p = \sum_{k=0}^p (-1)^{p-k} [P]_k \cdot n^k. \text{ So } [n]_{p-1} = \sum_{k=0}^{p-1} (-1)^{p-1-k} [P-1]_k \cdot n^k.$$

$$\text{Now } [n]_p = n(n-1)(n-2)\dots(n-(p-2)) \cdot [n-(p-1)] = [n]_{p-1} \cdot [n-(p-1)].$$

$$\text{So } [n]_p = [n-(p-1)] \cdot \sum_{k=0}^{p-1} (-1)^{p-1-k} [P-1]_k \cdot n^k$$

$$= \sum_{k=0}^{p-1} (-1)^{p-1-k} [P-1]_k \cdot n^{k+1} - (p-1) \cdot \sum_{k=0}^{p-1} (-1)^{p-1-k} [P-1]_k n^k$$

$$\text{replace } k \text{ by } k-1 \quad = \sum_{k=0}^{p-1} (-1)^{p-1-(k-1)} [P-1]_k n^{(k-1)+1} + (p-1) \sum_{k=0}^{p-1} (-1)^{p-1-k} [P-1]_k n^k$$

$$= \sum_{k=1}^p (-1)^{p-k} [P-1]_{k-1} n^k + \sum_{k=0}^{p-1} (-1)^{p-k} \cdot (p-1) \cdot [P-1]_k n^k$$

$$(**) = (-1)^{p-p} [P-1]_{p-1} n^p + \sum_{k=1}^{p-1} (-1)^{p-k} \cdot \left\{ [P-1]_{k-1} + (p-1) \cdot [P-1]_k \right\} \cdot n^k + (p-1) \cdot (-1)^p \cdot [P-1]_0 n^0$$

Comparing the coefficients of  $n^k$  in (\*) & (\*\*) for  $1 \leq k \leq p-1$

we get  $\begin{bmatrix} P \\ k \end{bmatrix} = \begin{bmatrix} p-1 \\ k-1 \end{bmatrix} + (p-1) \cdot \begin{bmatrix} p-1 \\ k \end{bmatrix}$ . END

[We also get  $\begin{bmatrix} P \\ p-1 \end{bmatrix} = \begin{bmatrix} p-1 \\ p-1 \end{bmatrix}$  &  $\begin{bmatrix} P \\ 0 \end{bmatrix} = (p-1) \begin{bmatrix} p-1 \\ 0 \end{bmatrix}$  — but this was not requested]