

*Answer all 6 questions. No Calculators, class-notes, or formula-sheets are allowed. An unjustified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.*

- (16) 1.(a) How many *integers* in  $\{1, 2, 3, \dots, 1000\}$  are *divisible* by *neither* 25 nor 30?  
 (b) Find the *permutation* which has  $\langle 3, 1, 2, 1, 1, 0 \rangle$  as its *inversion sequence* and check that your answer is correct.
- (16) 2.(a) Write down what is the *Multinomial Theorem* in full details. Then use it to find the *coefficient* of  $x^3y^5z^2$  in the expansion of  $(xy - 3y^2 + 2z)^6$ .  
 (b) Find the number of *integer solutions* of the equation:  $x_1 + x_2 + x_3 + x_4 = 9$  with  $x_1 \geq 2, x_2 \geq -1, x_3 \geq -3, \text{ and } x_4 \geq 4$ .
- (16) 3.(a) Write down what is the *first version* of the *Inclusion-Exclusion Theorem*.  
 (b) Find the number of *16-combinations* of the multi-set  $F = [5.a, 7.b, 18.c]$ .  
 [*You should leave your answer in terms of simplified Binomial coefficients.* ]
- (16) 4. (a) How many permutations of  $\{2, 3, 4, 5, 6, 7, 8\}$  have *each odd number* going to an *odd number and no even number* going to itself?  
 (b) How many permutations of  $\{2, 3, 4, 5, 6, 7, 8\}$  have *no composite number* going to itself? [*You may leave your answer in terms of simplified factorials.* ]
- (16) 5.(a) Let  $n \geq 0$ . Using the Binomial theorem, give an *analytic proof* of the fact:  

$$\binom{2n+1}{1} + \binom{2n+1}{3} + \binom{2n+1}{5} + \dots + \binom{2n+1}{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{2} + \binom{2n+1}{4} + \dots + \binom{2n+1}{2n}.$$
  
 (b) Give a *combinatorial proof* of the same result in part (a). Also write down what is *Newton's Binomial Theorem*.
- (20) 6.(a) Define what is a *positive set* with respect to the subsets  $A_1, A_2, \dots, A_n$  of  $U$ .  
 (b) Let  $n > 1$ . Find the value of the following sums (in simplified form).  
 (i)  $1 \cdot \binom{n}{0} - 2 \cdot \binom{n}{1} + 3 \cdot \binom{n}{2} - \dots + (-1)^n \cdot (n+1) \cdot \binom{n}{n} = (?)$   
 (ii)  $\frac{1}{1} \cdot \binom{n}{0} - \frac{1}{2} \cdot \binom{n}{1} + \frac{1}{3} \cdot \binom{n}{2} - \dots + (-1)^n \cdot \frac{1}{(n+1)} \cdot \binom{n}{n} = (?)$  END.

1(a) Let  $U = \{1, 2, 3, \dots, 1000\}$  &  $A_k = \{x \in U : x \text{ is divisible by } k\}$ . Then

$$A_{25} \cap A_{30} = A_{150} \text{ and our answer is } |A_{25}^c \cap A_{30}^c| = |U| - |A_{25}| - |A_{30}| + |A_{150}|$$

$$= 1000 - \left\lfloor \frac{1000}{25} \right\rfloor - \left\lfloor \frac{1000}{30} \right\rfloor + \left\lfloor \frac{1000}{150} \right\rfloor = 1000 - 40 - 33 + 6 = 933.$$

(b) Write down  $\langle 6 \rangle$  because the inversion sequence has length 6,

$$\begin{array}{l} \text{Then } \langle 6, 5 \rangle \text{ b.c. } i_5 = 1, \\ \text{then } \langle 6, 4, 5 \rangle \text{ b.c. } i_4 = 1, \\ \text{then } \langle 6, 4, 3, 5 \rangle \text{ b.c. } i_3 = 2. \end{array} \quad \begin{array}{l} \text{Then } \langle 6, 2, 4, 3, 5 \rangle \text{ b.c. } i_2 = 2, \\ \text{and finally } \langle 6, 2, 4, 1, 3, 5 \rangle \text{ b.c. } i_1 = 3. \\ \text{Check: inv. seg} = \langle 3, 1, 2, 1, 1, 0 \rangle \quad \checkmark \end{array}$$

2(a) For any  $k, n \in \mathbb{N}$ ,  $(x_1 + \dots + x_k)^n = \sum_{\{(n_1, \dots, n_k) : n_1 + n_2 + \dots + n_k = n\}} (n_1, \dots, n_k) x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$

$$\text{where } (n_1, n_2, \dots, n_k) = \frac{n!}{n_1! n_2! \dots n_k!}.$$

(b) Coeff. of  $x^3 y^5 z^2$  in the expansion of  $(xy - 3y^2 + 2z)^6$

$$= \text{coefficient of only the term from } (xy)^3, (-3y^2), (2z)^2$$

$$= \text{coeff. of } \binom{6}{3, 1, 2} \cdot (xy)^3 \cdot (-3y^2) \cdot (2z)^2 = \frac{6!}{3! 1! 2!} (-3) \cdot (2)^2 = \frac{6 \cdot 5 \cdot 4 \cdot (-12)}{2} = -720$$

(b) Put  $x_1 = y_1 + 2$ ,  $x_2 = y_2 - 1$ ,  $x_3 = y_3 - 3$ , and  $x_4 = y_4 + 4$ .

$$\text{So } (y_1 + 2) + (y_2 - 1) + (y_3 - 3) + (y_4 + 4) = 9 \text{ with } y_i \geq 0.$$

So  $y_1 + y_2 + y_3 + y_4 = 9 - 2 - 4 + 1 + 3 = 7$ , with  $y_i \geq 0$ . So the no. of solutions of  $x_1 + x_2 + x_3 + x_4 = 9$  with  $x_1 \geq 2$ ,  $x_2 \geq -1$ ,  $x_3 \geq -3$ ,  $x_4 \geq 4$

= no. of solutions in non-neg. integers of  $y_1 + y_2 + y_3 + y_4 = 7$

$$\text{which is } \binom{7+4-1}{4-1} = \binom{10}{3} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{3! 7!} = 120$$

3(a) Let  $A_1, \dots, A_n$  be subsets of a universal set  $U$ . Then version 1 of the I.E.P. says:  $|A_1^c \cap A_2^c \cap \dots \cap A_n^c| = \sum_{k=0}^n (-1)^k \{ \sum [\cup_{1 \leq i_1 < i_2 < \dots < i_k} A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}] \}$  where

The 2nd summation is over all the strictly incr. subseq. of  $\langle 1, 2, \dots, n \rangle$ .

(b) Let  $M = [\infty, a, \infty, b, \infty, c]$  and  $U = \text{set of all 16-comb. of } M$ . Then put

$A = \text{set of 16-comb. of } M \text{ with } \geq 6a's$ , so  $|A| = |\text{set of all 10-comb. of } M|$

$B = \text{set of 16-comb. of } M \text{ with } \geq 8b's$ , so  $|B| = |\text{set of all 8-comb. of } M|$

$C = \text{set of 16-comb. of } M \text{ with } \geq 19c's$ , so  $|C| = |\emptyset| = 0$ .

3(b) Also  $A \cap B = \text{set of } 16\text{-comb. of } M \text{ with } \geq 6a's \& \geq 8b's$ . Then

$$|A \cap B| = |\text{set of all 2-comb. of } M|, A \cap C = \emptyset, B \cap C = \emptyset, \& A \cap B \cap C = \emptyset.$$

So no. of 16-comb. of  $F = [5.a, 7.b, 18.c]$  is  $|A^c \cap B^c \cap C^c| =$

$$\begin{aligned} & |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| \\ &= \binom{16+3-1}{3-1} - \binom{10+3-1}{3-1} - \binom{8+3-1}{3-1} - 0 + \binom{2+3-1}{3-1} + 0 + 0 - 0 = \binom{18}{2} - \binom{12}{2} - \binom{10}{2} + \binom{4}{2}. \end{aligned}$$

4(a) Let  $D = \{3, 5, 7\}$  and  $E = \{2, 4, 6, 8\}$ . Then there are  $3!$  ways of permuting the elements of  $D$  and  $D_4 = 9$  ways of deranging the elements of  $E$ . (Note: since  $D \rightarrow D$ ,  $E$  must also go to  $E$ ). So no. of perm. of  $U = \{2, 3, \dots, 8\}$  with odd  $\rightarrow$  odd & no even going to itself  $= (3!) \cdot D_4 = 6(9) = \boxed{54}$ .

(b) Let  $C = \{4, 6, 8\}$ , the composite numbers in  $U$ . Put  $A_i = \text{set of all perm. of } U \text{ with } i \text{ going to itself}$ . Then  $|U| = 7!$ ,  $|A_i| = 6!$  for  $i = 4, 6 \& 8$ ,  $|A_i \cap A_j| = 5!$  for  $i \neq j$  and  $|A_4 \cap A_6 \cap A_8| = 4!$ . So our answer  $= |A_4^c \cap A_6^c \cap A_8^c| = |U| - |A_4| - |A_6| - |A_8| + |A_4 \cap A_6| + |A_4 \cap A_8| + |A_6 \cap A_8| - |A_4 \cap A_6 \cap A_8| = 7! - 3(6!) + 3(5!) - (4!).$

5(a) We know that  $(1+x)^{2n+1} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} x^k$ . Putting  $x = -1$  gives  $(1-1)^{2n+1} = 0 = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k = \sum_{k=0, k \text{ even}}^{2n+1} \binom{2n+1}{k} \cdot 1 + \sum_{k=1, k \text{ odd}}^{2n+1} \binom{2n+1}{k} \cdot (-1)$ . Hence  $\sum_{k=0, k \text{ even}}^{2n} \binom{2n+1}{k} = \sum_{k=1, k \text{ odd}}^{2n+1} \binom{2n+1}{k}$ .

$$\text{Thus } \binom{2n+1}{1} + \binom{2n+1}{3} + \dots + \binom{2n+1}{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{2} + \dots + \binom{2n+1}{2n}$$

(b) Let  $E = \text{set of all } k\text{-subsets of } \{1, 2, 3, \dots, 2n+1\} \text{ with } k \text{ even}$  and  $D = \text{set of all } k\text{-subsets of } \{1, 2, 3, \dots, 2n+1\} \text{ with } k \text{ odd}$ .

Define  $f: D \rightarrow E$  by  $f(S) = S - \{1\}$  if  $1 \in S$ , and  $S \cup \{1\}$  if  $1 \notin S$ , and  $g: E \rightarrow D$  by  $g(S) = S - \{1\}$  if  $1 \in S$ , and  $S \cup \{1\}$  if  $1 \notin S$ .

Then  $f \circ g: E \rightarrow E$  &  $g \circ f: D \rightarrow D$  will be  $\text{id}_E$  &  $\text{id}_D$ , the identity functions on  $E$  &  $D$  respectively. So  $g = f^{-1}$  & hence  $f: D \rightarrow E$  will be a bijection. Hence  $|D| = |E|$ . So we now get

$$\binom{2n+1}{1} + \binom{2n+1}{3} + \cdots + \binom{2n+1}{2n+1} = |D| = |E| = \binom{2n+1}{0} + \binom{2n+1}{2} + \cdots + \binom{2n+1}{2n}.$$

5(c) Newton's Binomial Theorem: Let  $\alpha \in \mathbb{R}$  and  $|x| < 1$ . Then

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \text{ converges and } \text{val}\left(\sum_{k=0}^{\infty} \binom{\alpha}{k} \cdot x^k\right) = (1+x)^{\alpha} \text{ where}$$

$$\binom{\alpha}{k} = \begin{cases} \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(k-1))/k! & \text{if } k \in \mathbb{Z}^+ \\ 1 \text{ (if } k=0\text{)}; \text{ and } 0, \text{ if } k \in \mathbb{Z}^- \end{cases}$$

In otherwords, for  $|x| < 1$  we have

$$(1+x)^{\alpha} = 1 + \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 + \cdots + \binom{\alpha}{k} x^k + \cdots$$

6(a) A positive set w.r.t. the subsets  $A_1, \dots, A_n$  of  $\mathcal{U}$  is any expression of the form  $\cup_{i_1} A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$  where  $\langle i_1, i_2, \dots, i_k \rangle$  is a strictly increasing subsequence of  $\langle 1, 2, 3, \dots, n \rangle$ .

$$(b) \text{ We know } x \cdot (1+x)^n = x \cdot \left[ \binom{n}{0} \cdot x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n} x^n \right]$$

$$\text{Differentiating gives } = \binom{n}{0} \cdot x^1 + \binom{n}{1} \cdot x^2 + \cdots + \binom{n}{n} \cdot x^{n+1}$$

$$1 \cdot (1+x)^n + x \cdot n \cdot (1+x)^{n-1} = 1 \cdot \binom{n}{0} \cdot x^0 + 2 \cdot \binom{n}{1} x^1 + 3 \cdot \binom{n}{2} x^2 + \cdots + (n+1) \cdot \binom{n}{n} x^n$$

$$\text{Putting } x=-1, \text{ gives us } 0 = \binom{n}{0} - 2 \cdot \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \cdot (n+1) \cdot \binom{n}{n}.$$

$$\text{We also know } (1+x)^n = \binom{n}{0} \cdot x^0 + \binom{n}{1} \cdot x^1 + \binom{n}{2} \cdot x^2 + \cdots + \binom{n}{n} \cdot x^n$$

Integrating both sides from  $-1$  to  $0$  gives us

$$\int_{-1}^0 (1+x)^n dx = \int_{-1}^0 \left\{ \binom{n}{0} \cdot x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n} \cdot x^n \right\} dx$$

$$\therefore \left[ \frac{(1+x)^{n+1}}{n+1} \right]_{-1}^0 = \left[ \binom{n}{0} \cdot \frac{x^1}{1} + \binom{n}{1} \frac{x^2}{2} + \binom{n}{2} \frac{x^3}{3} + \cdots + \binom{n}{n} \cdot \frac{x^{n+1}}{n+1} \right]_{-1}^0$$

$$\therefore \left[ \frac{1}{n+1} - 0 \right] = \left[ \binom{0}{0} - \left[ \frac{(-1)^1}{1} \binom{n}{1} + \frac{(-1)^2}{2} \binom{n}{2} + \frac{(-1)^3}{3} \binom{n}{3} + \cdots + \frac{(-1)^{n+1}}{n+1} \binom{n}{n} \right] \right]$$

$$\therefore \frac{1}{n+1} = \frac{1}{0} \binom{n}{0} - \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} - \cdots + \frac{(-1)^n}{n+1} \binom{n}{n} \quad \text{END}$$