

Answer all 6 questions. No calculators or class-notes are allowed. An unjustified answer will receive little credit. Begin each of the 6 questions on 6 separate pages.

- (16) 1. Find the solution of the recurrence equation $x_{n+2} + x_{n+1} - 12x_n = 0$ with the initial conditions $x_0 = 3$ and $x_1 = 2$.
- (18) 2. Find the complementary solution and give (with reasons) the minimum form of a particular solution of each of the following recurrence equations.
(a) $(E^2 + 5I)(E^2 + 3E - 4I)(x_n) = 2n^3 + n$.
(b) $(E^2 - 5I)(E^2 - 2E + 5I)(x_n) = 4n^2 \cdot (5)^{n/2}$.
- (18) 3.(a) Use the method of generating functions to find the solution of the recurrence equation $a_n - 3a_{n-1} + 4 = 0$ with the initial condition $a_0 = 3$.
(b) Let $M = [9.a, \infty.b, \infty.c]$. Find the generating function of the collection of all n -combinations of M with an even number of a's, an odd number of b's, and at least 2 c's. [Simplify your answers as far as possible.]
- (16) 4.(a) Define what are the operators E and Δ that operates on a sequence $\langle h_k \rangle_{k \in \mathbb{N}}$.
(b) Let $h_k = 3k^2 - 2k + 5$. Find the zero-column of $\langle h_k \rangle_{k \in \mathbb{N}}$ and use it to get a formula for the sum $h_0 + h_1 + h_2 + \dots + h_n$. [Simplify answer as far as possible.]
- (16) 5.(a) Write down what is the function form of the Pigeon-hole Principle.
(b) Let n be a positive integer & $s = \langle a_0, a_1, a_2, \dots, a_{3n} \rangle$ be any sequence of $3n+1$ integers. Prove that we can always find a non-empty segment (made up of consecutive terms) whose terms add up to a multiple of $3n$.
- (16) 6.(a) Define what are the Stirling coefficients of the 2nd kind and define what are the Stirling Partition numbers.
(b) Prove for any $k, p \in \mathbb{N}$ with $1 \leq k \leq p-1$, the Stirling Partition numbers satisfy $S(p, k) = S(p-1, k-1) + k \cdot S(p-1, k)$.

[In questions #5 and #6, you must prove everything by using the definitions. You are not allowed to use any similar-looking theorem proved in class.]

1. We have $x_{n+2} + x_{n+1} - 12x_n = 0$. So $(E^2 + E - 12I)(X_n) = 0$.

$\therefore (E-3I)(E+4I)x_n = 0$. Roots are 3 & -4. So $x_n = A.(3)^n + B.(-4)^n$

$\therefore x_0 = 3 = A+B$ & $x_1 = 2 = A(3) + B(-4)$ So $B = 3-A$ and hence

$$2 = 3A + (3-A)(-4) \Rightarrow 2 = 7A - 12 \Rightarrow 7A = 14 \Rightarrow A = 2. \therefore B = 3-2 = 1$$

So $x_n = 2.(3)^n + 1.(-4)^n$. Check: $2(3)^0 + (-4)^0 = 3$, $2(3)^1 + (-4)^1 = 2 \checkmark$

2.(a) $(E^2 + 5I)(E^2 + 3E - 4I) = 0 \Rightarrow (E - i\sqrt{5})(E + i\sqrt{5})(E + 4)(E - 1) = 0$. So

$E = i\sqrt{5}, -i\sqrt{5}, -4$, or 1 $\therefore x_n^c = A.(i\sqrt{5})^n + B.(-i\sqrt{5})^n + C.(-4)^n + D.(1)^n$.

Since $2n^3 + n$ is a polynomial of degree 3 & 1 is a root of the auxiliary equation of multiplicity 1, the minimal form of a particular solution will be $x_n^p = (a + bn + cn^2 + dn^3).n^1.(1)^n$

(b) $(E^2 - 5I)(E^2 - 2E + 5I) = 0 \Rightarrow (E - \sqrt{5}I)(E + \sqrt{5}I)[(E-1)^2 + 4] = 0$. So

$E = \sqrt{5}, -\sqrt{5}, 1+2i$, or $1-2i$, $\therefore x_n^c = A(1+2i)^n + B(1-2i)^n + C.(-\sqrt{5})^n + D.(\sqrt{5})^n$

Since $4n^2.(\sqrt{5})^{n/2} = 4n^2.(\sqrt{5})^n = (\text{a polynomial of deg. 2}).(\sqrt{5})^n$ and $\sqrt{5}$ is a root of the auxiliary equation of multiplicity 1, the minimal form of a particular sol. will be $x_n^p = (a + bn + cn^2).(\sqrt{5})^n$.

3(a) Let $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$

Then $-3x.f(x) = \dots -3a_0x - 3a_1x^2 - \dots -3a_{n-1}x^n - \dots$

and $4/(1-x) = 4 + 4x + 4x^2 + \dots + 4x^n + \dots$

$$\begin{aligned} (1-3x)f(x) + 4/(1-x) &= (a_0 + 4) + (a_1 - 3a_0 + 4)x + \dots + (a_n - 3a_{n-1} + 4)x^n + \dots \\ &= (3+4) + 0x + \dots + 0x^n + \dots = 7. \end{aligned}$$

$$\therefore (1-3x)f(x) = 7 - \frac{4}{1-x} = \frac{(3-7x)}{1-x} \Rightarrow f(x) = \frac{3-7x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$\therefore 3-7x = A(1-3x) + B(1-x)$. Putting $x=1$, gives $-4 = A(-2) \Rightarrow A = 2$.

Putting $x=1/3$, gives $3 - (7/3) = B(2/3) \Rightarrow 2/3 = (2/3)B \Rightarrow B = 1$.

$\therefore f(x) = \frac{2}{1-x} + \frac{1}{1-3x}$. So $x_n = \text{coeff. of } x^n \text{ in the expansion}$

$$2[1+x+x^2+\dots+x^n+\dots] + 1.[1+3x+(3x)^2+\dots+(3x)^n+\dots] = 2 + 1.(3)^n.$$

3(b) The generating function of the n -combinations of M

$$\begin{aligned}
 &= (x^0 + x^1 + \dots + x^8)(x^1 + x^3 + x^5 + x^7 + \dots)(x^2 + x^3 + x^4 + x^5 + \dots) \\
 &= x^1 \cdot x^2 \cdot (1 + x^2 + (x^2)^2 + (x^2)^3 + (x^2)^4) \cdot (1 + x^2 + x^4 + \dots) (1 + x^2 + x^4 + \dots) \\
 &= x^3 \frac{[1 - (x^2)^{4+1}]}{1 - x^2} \cdot \frac{1}{1 - x^2} \cdot \frac{1}{1 - x} = \frac{x^3(1 - x^{10})}{(1 - x^2)^2(1 - x)} = \frac{x^3 \cdot (1 - x^{10})}{(1 - x)^3 \cdot (1 + x)^2}.
 \end{aligned}$$

4(a) $E(\langle X_n \rangle_{n \in \mathbb{N}}) = \langle X_{n+1} \rangle_{n \in \mathbb{N}}$ $\Delta(\langle X_n \rangle_{n \in \mathbb{N}}) = \langle X_{n+1} - X_n \rangle_{n \in \mathbb{N}}$.

(b)

0	1	2	3	4	5
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$$\langle \Delta^0 h_k \rangle \quad 5 \quad 6 \quad 13 \quad 26 \quad 45 \quad 70 \quad \leftarrow \langle h_k \rangle$$

$$\langle \Delta^1 h_k \rangle \quad 1 \quad 7 \quad 13 \quad 19 \quad 25 \quad .$$

$$\langle \Delta^2 h_k \rangle \quad 6 \quad 6 \quad 6 \quad 6 \quad . \quad .$$

$$\langle \Delta^3 h_k \rangle \quad 0 \quad 0 \quad 0 \quad . \quad . \quad .$$

\therefore Zero column $= \langle 5, 1, 6, 0, 0, 0, \dots \rangle$ because $\deg(h_k) = 2$.

$$\begin{aligned}
 \text{Also } h_k &= 2 \binom{k}{0} + 1 \cdot \binom{k}{1} + 6 \binom{k}{2} \quad \& \sum_{k=0}^n h_k &= 2 \cdot \binom{n+1}{1} + 1 \cdot \binom{n+1}{2} + 6 \cdot \binom{n+1}{3} \\
 &= 2(n+1) + (n+1)(n)/2 + 6 \cdot (n+1)(n)(n-1)/6 = (n+1)[2n^2 - n + 5]/2.
 \end{aligned}$$

5(a) Let $f: P \rightarrow H$ be a function & suppose that P & H are finite sets with $|P| > |H| > 0$. Then there exists $x_1, x_2 \in P$ with $x_1 \neq x_2$ & $f(x_1) = f(x_2)$.

(b) Let $P = \text{set of sums of the initial segments of } S$ (including $\langle \rangle$)

$$= \left\{ \sum_{k=0}^i a_i \right\}_{i=1 \text{ to } 3n} = \{0, a_0, (a_0 + a_1), (a_0 + a_1 + a_2), \dots, (a_0 + a_1 + \dots + a_{3n})\}$$

Then $|P| = 3n+2$. Also let $H = \{0, 1, 2, 3, \dots, 3n-1\}$, Then $|H| = 3n$.

Now define $f: P \rightarrow H$ by $f(x) = x \pmod{3n}$. Then by the function form of the Pigeon-Hole Principle, we can find i and j with $-1 \leq i < j \leq 3n$ such that $\sum_{k=0}^i a_k = \sum_{k=0}^j a_k$. So we will get

$$a_0 + a_1 + a_2 + \dots + a_i = a_0 + a_1 + a_2 + \dots + a_i + \dots + a_j \pmod{3n}$$

Hence $a_{i+1} + a_{i+2} + a_{i+3} + \dots + a_j \equiv 0 \pmod{3n}$

$\therefore \langle a_{i+1}, a_{i+2}, a_{i+3}, \dots, a_j \rangle$ will be a non-empty segment (because $i < j$) whose terms add up to a multiple of $3n$.

6(a) The Stirling coefficients of the 2nd kind are the unique integers $\{S_k^p\}$ such that $n^p = \sum_{k=0}^p \{S_k^p\} \cdot [n]_k$ where $[n]_k = n(n-1)(n-2)\dots(n-(k-1)) = n!/(n-k)!$

The Stirling Partition numbers $S(p,k)$ is defined by $S(p,k) = \text{number of partitions of } \{1, 2, 3, \dots, p\} \text{ into } k \text{ parts.}$

(b) Let $A = \text{the set of all partitions of } \{1, 2, 3, \dots, p\} \text{ into } k \text{ parts}$,
 $B = \text{set of partitions in } A \text{ with } p \text{ being in a part by itself}$,
 $\& B_0 = \text{set of partitions in } A \text{ with } p \text{ not in a part by itself}$.
 Then $B \cap B_0 = \emptyset$ and $A = B \cup B_0$. So $|A| = |B| + |B_0|$.

Now if we remove the part $\{p\}$ from a partition in B , then we will get a partition of $\{1, 2, 3, \dots, p-1\}$ into $(p-1)$ parts. And if we add the part $\{p\}$ to a partition of $\{1, 2, 3, \dots, p-1\}$ into $(p-1)$ parts, we will get a partition of B . So $|B| = S(p-1, k-1)$

Also if we take a partition of $\{1, 2, 3, \dots, p-1\}$ into k parts and add p to each of the k parts in turns, we will get k partitions of B_0 . Moreover, if we remove p from its respective part in a partition of B_0 , we will get a partition of $\{1, 2, 3, \dots, p-1\}$ into k parts. So $|B_0| = S(p-1, k)$. Hence

$$S(p, k) = |A| = |B| + |B_0| = S(p-1, k-1) + k \cdot S(p-1, k)$$

for all $1 \leq k \leq p-1$.