

Answer all 6 questions. No calculators, cell-phones, or class-notes are allowed. An un-justified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.

- (16) 1.(a) How many *integers* in $\{1, 2, 3, \dots, 1000\}$ are *divisible* by *neither* 16 nor 20?
(b) Find the *permutation* which has $\langle 2, 1, 3, 1, 1, 0 \rangle$ as its *inversion sequence* and check that your answer is correct.
- (16) 2.(a) Write down what is the *Multinomial Theorem* in full details. Then use it to find the *coefficient* of $x^5y^2z^3$ in the expansion of $(x^2 + 3y - 2xz)^6$.
(b) Find the number of *integer solutions* of the equation: $x_1 + x_2 + x_3 + x_4 = 11$ with $x_1 \geq -1$, $x_2 \geq -3$, $x_3 \geq 2$, and $x_4 \geq 5$.
- (16) 3.(a) Write down what is the *first version* of the *Inclusion-Exclusion Theorem*.
(b) Find the number of *17-combinations* of the multi-set $F = [7.a, 5.b, 19.c]$.
[You should leave your answer in terms of *simplified Binomial coefficients*.]
- (16) 4. (a) How many permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ have *each even number* going to an *even number* and *no odd number* going to itself?
(b) How many permutations of $\{2, 3, 4, 5, 6, 7, 8\}$ have *no odd number* going to itself? [You may leave your answer in terms of *simplified factorials*.]
- (16) 5.(a) Let $n \geq 1$, n *even*. Using the Binomial theorem, give an *analytic proof* of
$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n}.$$

(b) Write down what is Pascal's identity and use it to prove that for $0 \leq k \leq n$,
$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}.$$
- (20) 6.(a) Define what are *positive sets* & *ultimate sets* w.r.t the subsets A_1, \dots, A_n of U .
(b) Let $n \geq 0$. Find the value of the following sums (*in simplified form*).
(i) $2 \cdot \binom{n}{0} + 3 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} + \dots + (n+2) \cdot \binom{n}{n} = (?)$.
(ii) $\frac{1}{1} \cdot \binom{n}{0} + \frac{1}{2} \cdot \binom{n}{1} + \frac{1}{3} \cdot \binom{n}{2} + \dots + \frac{1}{(n+1)} \cdot \binom{n}{n} = (??)$. END.

3(b) Also $(A \cap B) =$ set of 17-comb. of M with ≥ 8 's & ≥ 6 's.

So $|A \cap B| = |\text{set of all 3-comb. of } M|$, $A \cap C = \emptyset$, $B \cap C = \emptyset$ & $A \cap B \cap C = \emptyset$.

\therefore No. of 16-comb. of $F = [7a, 5b, 19c]$ is $|A^c \cap B^c \cap C^c| =$

$$\begin{aligned} & (21) - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| \\ &= \binom{17+3-1}{3-1} - \binom{9+3-1}{3-1} - \binom{11+3-1}{3-1} - 0 + \binom{3+3-1}{3-1} + 0 + 0 - 0 = \binom{19}{2} - \binom{11}{2} - \binom{13}{2} + \binom{5}{2}. \end{aligned}$$

4(a) Let $E = \{2, 4, 6, 8\}$ & $D = \{1, 3, 5, 7\}$. Then there are $4!$ ways of permuting the elements of E & $\mathcal{D}_4 = 9$ ways of deranging the elements of D . [Note: Since $E \rightarrow E$, D must also go to D .] So no. of permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ with even \rightarrow even & no odd going to itself

$$= (4!) \mathcal{D}_4 = (24)(9) = \boxed{216}.$$

(b) Let $C = \{3, 5, 7\}$. Put $A_i =$ set of all perm. of $\{2, 3, 4, 5, 6, 7, 8\}$ with i going to itself and $U =$ set of all perm. of $\{2, 3, 4, 5, 6, 7, 8\}$.

$$\begin{aligned} \text{Then our answer will be } & |A_3^c \cap A_5^c \cap A_7^c| = |U| - |A_3| - |A_5| \\ & - |A_7| + |A_3 \cap A_5| + |A_3 \cap A_7| + |A_5 \cap A_7| - |A_3 \cap A_5 \cap A_7| = 7! - 3(6!) + 3(5!) - 4! \end{aligned}$$

5(a) We know that $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$.

Putting $x = -1$, gives $(1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots - \binom{n}{n-1} + \binom{n}{n}$.

$$\text{So } \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n}.$$

(b) Pascal's identity is: $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ for $0 \leq k \leq n$

We will prove $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}$ by parametric induction on n (k will be the parameter)

Basis: For $n=k$, we have $\binom{k}{k} = 1 = \binom{k+1}{k+1}$. So the result is true for $n=k$.

Ind. step: Suppose $n \geq k$ and the result is true for n . Then

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}.$$

$$\begin{aligned} \text{So } & \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} + \binom{n+1}{k} = \binom{n+1}{k+1} + \binom{n+1}{k} \\ & = \binom{(n+1)+1}{k+1} \text{ by Pascal's identity.} \end{aligned}$$

So if the result is true for n , it will be true for $n+1$.

Conclusion: So the result is true for all $0 \leq k \leq n$ by Parametric Induction

6(a) A positive set w.r.t. the subsets A_1, A_2, \dots, A_n of U is any expression of the form $U \cap A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}$ where $\langle i_1, i_2, \dots, i_k \rangle$ is a subsequence of $\langle 1, 2, 3, \dots, n \rangle$. An ultimate set w.r.t. the subsets A_1, A_2, \dots, A_n of U is any expression of the form $X_1 \cap X_2 \cap \dots \cap X_n$ where $X_i = A_i$ or A_i^c for $i = 1, 2, 3, \dots, n$.

(b)(i) We know $(1+x)^n = \binom{n}{0} \cdot x^0 + \binom{n}{1} \cdot x^1 + \binom{n}{2} \cdot x^2 + \dots + \binom{n}{n} \cdot x^n$.
So $\binom{n}{1} \cdot x^2 + \binom{n}{1} \cdot x^3 + \binom{n}{2} \cdot x^3 + \dots + \binom{n}{n} \cdot x^{n+2} = x^2 \cdot (1+x)^n$.

Differentiating both sides w.r.t. x gives us

$$2 \cdot \binom{n}{0} x^1 + 3 \cdot \binom{n}{1} x^2 + 4 \cdot \binom{n}{2} x^2 + \dots + (n+2) \cdot \binom{n}{n} x^{n+1} \\ = \frac{d}{dx} [x^2 \cdot (1+x)^n] = 2x \cdot (1+x)^n + x^2 \cdot n \cdot (1+x)^{n-1}$$

Putting $x=1$ gives

$$2 \binom{n}{0} + 3 \binom{n}{1} + 4 \binom{n}{2} + \dots + (n+2) \binom{n}{n} = 2 \cdot 2^n + n \cdot 2^{n-1} \\ = (4+n) \cdot 2^{n-1}$$

(ii) We know $\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n = (1+x)^n$

Integrating both sides from 0 to 1 gives us

$$\int_0^1 \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n \right] dx = \int_0^1 (1+x)^n dx$$

$$\therefore \left[\frac{1}{1} \binom{n}{0}x + \frac{1}{2} \binom{n}{1}x^2 + \frac{1}{3} \binom{n}{2}x^3 + \dots + \frac{1}{(n+1)} \binom{n}{n}x^{n+1} \right]_0^1 = \left[\frac{(1+x)^{n+1}}{n+1} \right]_0^1$$

Put $u = 1+x$, $du = dx$

$$\int (1+x)^n dx = \int u^n du = \frac{u^{n+1}}{n+1} + C = \frac{(1+x)^{n+1}}{n+1}$$

$$\therefore \frac{1}{1} \binom{n}{0} + \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} + \dots + \frac{1}{n+1} \binom{n}{n} = \frac{2^{n+1} - 1}{n+1} \quad \text{END}$$