

Answer all 6 questions. No calculators or class-notes are allowed. An unjustified answer will receive little credit. Begin each of the 6 questions on 6 separate pages.

- (15) 1. Find *the solution* of the recurrence equation $x_{n+1} - 2x_n = 2 \cdot (3)^n$ with the *initial condition* $x_0 = 5$.
- (20) 2. Find the *complementary solution* and give (with reasons) the *minimum form* of a *particular solution* of each of the following *recurrence equations*.
- (a) $(E^2 + I)^2 (E^2 + 4E - 5I) (x_n) = 5n$.
(b) $(E^2 - 8I) (E^2 - 2E + 4I) (x_n) = n \cdot (2)^{3n/2}$.
- (20) 3.(a) Use the *method of generating functions* to find the solution of the recurrence equation $a_n + a_{n-1} - 2a_{n-2} = 0$ with the initial condition $a_0 = 5$ and $a_1 = -4$.
(b) Let $M = [\infty.a, \infty.b, 12.c]$. Find the *generating function* of the collection of all n -combinations of M with at least 4 a's, an odd number of b's, and an even number of c's. [Simplify your answers as far as possible.]
- (15) 4.(a) Define what is the operator Δ & what is the *zero column* of a sequence $\langle h_k \rangle_{k \in \mathbb{N}}$.
(b) Let $h_k = 3k^2 - k + 1$. Find the *zero-column* of $\langle h_k \rangle_{k \in \mathbb{N}}$ and use it to get a formula for the sum $h_0 + h_1 + h_2 + \dots + h_n$. [Simplify answer as far as possible.]
- (15) 5.(a) Write down what is the *function form* of the Pigeon-hole Principle.
(b) Let n be a positive integer and $s = \langle a_1, a_2, a_3, \dots, a_{3n} \rangle$ be any sequence of $3n$ integers. Prove that we can always find a *non-empty segment* (made up of consecutive terms) whose *terms* add up to a multiple of $3n$.
- (15) 6.(a) Define what are the *Stirling coefficients of the First kind & the Second kind*.
(b) Prove for any k with $1 \leq k \leq p-1$, the Stirling coefficients of the *Second kind* satisfy $\begin{Bmatrix} p \\ k \end{Bmatrix} = \begin{Bmatrix} p-1 \\ k-1 \end{Bmatrix} + k \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix}$.

[In questions #5 and #6, you must prove everything by using the definitions. You are **not allowed** to use any similar-looking theorem proved in class.]
[You should check your solutions ONLY after answering all the questions.]

1. The homogeneous eq. is $x_{n+1} - 2x_n = 0$. So $(E - 2I)x_n = 0$

$\therefore x_n^C = A \cdot (2)^n$. Since 2 is a polynomial of degree 0 and 3 is not a root of the aux. eq., we should try $x_n^P = b \cdot (3)^n$ as a particular solution. So

$$x_{n+1}^P - 2x_n^P = 2 \cdot (3)^n \text{ becomes } b \cdot (3)^{n+1} - 2 \cdot b \cdot (3)^n = 2 \cdot (3)^n.$$

$$\text{Hence } [(3b) - (2b)] \cdot (3)^n = 2 \cdot (3)^n \text{ & so } b = 2. \therefore x_n^P = 2 \cdot (3)^n.$$

$$\text{Thus } x_n = x_n^C + x_n^P = A \cdot (2)^n + 2 \cdot (3)^n. \text{ Now } x_0 = 5, \text{ so}$$

$$5 = A \cdot (2)^0 + 2 \cdot (3)^0 \text{ and thus } A + 2 = 5 \Rightarrow A = 3. \text{ Thus}$$

$$x_n = A \cdot (2)^n + 2 \cdot (3)^n = 3 \cdot (2)^n + 2 \cdot (3)^n. \quad [\text{Check: } x_0 = 3 \cdot (2)^0 + 2 \cdot (3)^0 = 5]$$

$$2.(a) (E^2 + I)^2 (E^2 + 4E - 5I)(x_n) = (5n + 0) \cdot (1)^n.$$

$$\text{Aux. eq. is } [(E - i)(E + i)]^2 [(E + 5)(E - 1)] = 0. \text{ Hence}$$

$$E = i, i, -i, -i, -5, \text{ and } 1. \text{ So}$$

$$x_n^C = (A + Bn) \cdot (i)^n + (C + Dn) \cdot (-i)^n + E \cdot (-5)^n + F \cdot (1)^n.$$

Since 1 is a root of the aux. eq. of multiplicity 1, and $5n + 0$ is a polynomial of degree 1, the minimal form of $x_n^P = (a + b \cdot n) \cdot n^1 \cdot (1)^n$.

$$(b) (E^2 - 8I)(E^2 - 2E + 4I)(x_n) = n \cdot (2)^{3n/2} = (1n + 0) \cdot (\sqrt{8})^n$$

$$\text{Aux. eq. is } (E - \sqrt{8})(E + \sqrt{8})[(E - 1)^2 + 3] = 0. \text{ Hence}$$

$$E = \sqrt{8}, -\sqrt{8}, 1 + i\sqrt{3}, 1 - i\sqrt{3}. \text{ So}$$

$$x_n^C = A \cdot (\sqrt{8})^n + B \cdot (-\sqrt{8})^n + C \cdot (1 + i\sqrt{3})^n + D \cdot (1 - i\sqrt{3})^n.$$

Since $\sqrt{8}$ is root of the aux. eq. of multiplicity 1, and $(1n + 0)$ is a polynomial of degree 1, the minimal form of x_n^P will be $x_n^P = (a + b \cdot n) \cdot n^1 \cdot (\sqrt{8})^n$

$$3(a) \text{ Let } f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

$$\text{Then } x \cdot f(x) = a_0 x + a_1 x^2 + \dots + a_{n-1} x^n + \dots$$

$$\text{and } -2 \cdot x^2 \cdot f(x) = -2a_0 x^2 - \dots - 2a_{n-2} x^n - \dots$$

(2)

$$3(a) \quad S_0 \quad (1+x-2x^2) f(x) = a_0 + (a_1+a_0)x + 0.x^2 + 0 + \dots + 0 + \dots$$

$$\text{Since } a_0 = 5 \text{ & } a_1 = -4, \quad (1+x-2x^2) = 5 + (5-4)x = 5+x.$$

$$\text{Hence } f(x) = \frac{5+x}{(1-x)(1+2x)} = \frac{A}{1-x} + \frac{B}{1+2x} = \frac{A(1+2x)+B(1-x)}{(1-x)(1+2x)}$$

$$\text{Thus } 5+x = A(1+2x) + B(1-x). \text{ Putting } x=1, \text{ gives us}$$

$$5+1 = A(1+2) \Rightarrow 3A = 6 \Rightarrow A = 2$$

$$\text{Putting } x=-\frac{1}{2}, \text{ gives us } 5-\frac{1}{2} = B(1+\frac{1}{2}) \Rightarrow \frac{9}{2} = \frac{3B}{2}$$

$$\text{So } B = 3. \text{ Hence } f(x) = \frac{2}{1-x} + \frac{3}{1+2x}$$

$$= 2[1+x+x^2+\dots+x^n+\dots] + 3[1+(-2x)+(-2x)^2+\dots+(-2x)^n]$$

$$\therefore a_n = \text{coefficient of } x^n \text{ in the expansion of } f(x)$$

$$= 2 \cdot (1)^n + 3 \cdot (-2)^n = 2 + 3 \cdot (-2)^n.$$

$$[\text{Quick check: } a_0 = 2 + 3(-2)^0 = 5 \checkmark, \quad a_1 = 2 + 3(-2)^1 = 4 \checkmark]$$

(b). The generating function of the n -comb. of $M = [\infty, a, \infty, b, 12, c]$

with at least 4 a's, an odd no. of b's, and an even no. of c's is

$$(x^4 + x^5 + x^6 + \dots) \cdot (x^1 + x^3 + x^5 + \dots) \cdot (x^0 + x^2 + x^4 + \dots + x^{12})$$

$$= x^4 (1+x+x^2+\dots) \cdot x \cdot [1+(x^2)+(x^2)^2+(x^2)^3+\dots], [1+(x^2)+(x^2)^2+\dots+(x^2)^6]$$

$$= x^4 \cdot (1-x)^{-1} \cdot x \cdot (1-x^2)^{-1} \cdot [1-(x^2)^{6+1}] / (1-x^2)$$

$$= \frac{x^5}{1-x} \cdot \frac{1}{(1-x)(1+x)} \cdot \frac{1-x^{14}}{(1-x)(1+x)} = \frac{x^5 \cdot (1-x^{14})}{(1-x)^3 \cdot (1+x)^2}$$

$$4(a) \quad \Delta(\langle h_k \rangle) = \langle h_{k+1} - h_k \rangle_{k \in \mathbb{N}} \quad \& \text{ zero-column} = \langle \Delta^k h_0 \rangle_{k \in \mathbb{N}}.$$

$$\text{Here } \Delta^0 \langle h_k \rangle = \langle h_k \rangle \quad \& \quad \Delta^{k+1} \langle h_n \rangle = \Delta(\Delta^k \langle h_n \rangle).$$

$$(b) \quad \begin{array}{c|ccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \Delta^0 \langle h_k \rangle & 1 & 3 & 11 & 25 & 45 & 71 & \dots \\ \Delta^1 \langle h_k \rangle & 2 & 8 & 14 & 20 & 26 & \dots & \\ \Delta^2 \langle h_k \rangle & 6 & 6 & 6 & 6 & \dots & & \\ \Delta^3 \langle h_k \rangle & 0 & 0 & 0 & \dots & & & \end{array}$$

\leftarrow zero diagonal

$$\text{So } h_k = 1 \cdot \binom{k}{0} + 2 \cdot \binom{k}{1} + 6 \cdot \binom{k}{2} + 0 + 0 + 0 \dots$$

$$\text{and } \sum_{k=0}^n h_k = 1 \cdot \binom{n+1}{1} + 2 \cdot \binom{n+1}{2} + 6 \cdot \binom{n+1}{3}$$

$$= (n+1) + 2 \cdot \binom{n+1}{2} \cdot (n) + 6 \cdot \binom{n+1}{3} \cdot (n-1)$$

$$= (n+1) [1+n+\binom{n}{2}(n-1)] = (n+1)(n^2+1).$$

(3)

5(a) Suppose $f: P \rightarrow H$ is a function where $P & H$ are finite non-empty sets. If $|P| > |H|$, then we can find $x_1, x_2 \in P$ with $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$.

(b) Let $P = \{0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + a_3 + \dots + a_{3n}\}$
 $= \left\{ \sum_{i=1}^k a_i : 0 \leq k \leq 3n \right\}$ and $H = \{0, 1, 2, \dots, 3n-1\}$. Define
 $f: P \rightarrow H$ by $f(x) = x \pmod{3n}$. Since $|P| = 3n+1 > 3n = |H|$, we can find $i & j$ with $0 \leq i < j \leq 3n$ such that $f(a_1 + a_2 + \dots + a_i) = f(a_1 + a_2 + \dots + a_j)$. So

$a_1 + a_2 + \dots + a_j \equiv a_1 + a_2 + \dots + a_i \pmod{3n}$, and so $a_{i+1} + a_{i+2} + \dots + a_j \equiv 0 \pmod{3n}$. Hence the segment $\langle a_{i+1}, a_{i+2}, \dots, a_j \rangle$ of $j-i$ consecutive terms will add up to a multiple of $3n$. So we can always find a non-empty segment of s whose terms add up to a multiple of $3n$.

6(a) The Stirling coeff. of the 1st kind are the unique integers $[P]_k$ such that $[n]_p = \sum_{k=0}^p (-1)^{p-k} [P]_k \cdot n^k$. The Stirling coeff. of the 2nd kind are the unique integers $\{P\}_k$ such that $n^p = \sum_{k=0}^p \{P\}_k \cdot [n]_k$. Here $[n]_p = n(n-1)(n-2)\dots[n-(k-1)]$.

(b) We know $n^{p-1} = \sum_{k=0}^{p-1} \{P-1\}_k \cdot [n]_k$ by replacing p by $(p-1)$ in the definition in part (a). So

$$\begin{aligned} n^p &= n^{p-1} \cdot (n) = \sum_{k=0}^{p-1} \{P-1\}_k \cdot [n]_k \cdot (n) = \sum_{k=0}^{p-1} \{P-1\}_k \cdot [n]_k \cdot [(n-k)+k] \\ &= \sum_{k=0}^{p-1} \{P-1\}_k \cdot [n]_k \cdot (n-k) + \sum_{k=0}^{p-1} k \cdot \{P-1\}_k \cdot [n]_k \\ &= \sum_{k=0}^{p-1} \{P-1\}_k \cdot [n]_{k+1} + \sum_{k=0}^{p-1} k \cdot \{P-1\}_k \cdot [n]_k = \sum_{k=1}^p \{P-1\}_k \cdot [n]_k + \sum_{k=0}^{p-1} k \cdot \{P-1\}_k \cdot [n]_k \\ &= \{P-1\}_p \cdot [n]_p + \sum_{k=1}^{p-1} (\{P-1\}_{k-1} + k \cdot \{P-1\}_k) \cdot [n]_k + 0 \cdot \{P-1\}_0 \cdot [n]_0 \end{aligned}$$

But $n^p = \{P\}_p \cdot [n]_p + \sum_{k=1}^{p-1} \{P\}_k \cdot [n]_k + \{P\}_0 \cdot [n]_0$ by definition

Hence $\{P\}_k = \{P-1\}_{k-1} + k \cdot \{P-1\}_k$ for each k with $1 \leq k \leq p$. END.