

*Answer all 6 questions. No Calculators or Cell phones are allowed. An unjustified answer will receive little credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.*

- (15) 1(a) How many *integers* in the set  $\{1, 2, 3, \dots, 800\}$  are divisible by 12 or 15?  
(b) Find the *permutation* which has  $\langle 4, 2, 3, 1, 1, 0 \rangle$  as its inversion sequence.
- (15) 2(a) Write down what the *Multinomial Theorem* says. Then find the coefficient of  $x^2yz^3$  in the expansion of  $(3x - 4y + 2z)^6$ .  
(b) Find the number of *integer solutions* of the equation  $x_1 + x_2 + x_3 = 10$  with  $x_1 \geq 1$ ,  $x_2 \geq -3$ , and  $x_3 \geq 4$ .
- (15) 3(a) How many permutations of  $\{0, 1, 2, \dots, 6\}$  have each *odd element* going to an *odd element* and *no even element* going to itself.  
(b) How many permutations of  $\{0, 1, 2, \dots, 6\}$  have no *odd element* going to itself ?  
[You may leave your answers in terms of simplified factorials.]
- (20) 4(a) Find the number of ways the letters of *SUCCESS* can be arranged in a line.  
(b) How many 15-combinations of the multi-set  $M = [5.a, 7.b, 18.c]$  are there ?  
[You may leave your answer in terms of simplified binomial coefficients.]
- (20) 5(a) Give a *combinatorial proof* of Pascal's identity  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$   
for all integers k and n with  $1 \leq k \leq n-1$ .  
(b) Hence, or otherwise, prove that for all integers k and n with  $0 \leq k \leq n$   
$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}.$$
- (15) 6(a) Write down the General form of the *Pigeon-Hole Principle*.  
(b) Let  $\langle a_1, a_2, \dots, a_n \rangle$  be a sequence of  $n$  integers. Prove that we can always find a *non-empty segment* of this sequence whose terms adds up to an integer multiple of  $n$ .

1(a) Let  $U = \{1, 2, 3, \dots, 800\}$ ,  $A = \{x \in U : x \text{ is divisible by } 12\}$  and  $B = \{x \in U : x \text{ is divisible by } 15\}$ . Then  $A \cap B = \{x \in U : x \text{ is divisible by } \text{lcm}(12, 15)\} = \{x \in U : x \text{ is divisible by } 60\}$ . So Answer =  $|A \cup B| = |A| + |B| - |A \cap B| = \left\lfloor \frac{800}{12} \right\rfloor + \left\lfloor \frac{800}{15} \right\rfloor - \left\lfloor \frac{800}{60} \right\rfloor$   
 $= \left\lfloor \frac{200}{3} \right\rfloor + \left\lfloor \frac{160}{3} \right\rfloor - \left\lfloor \frac{40}{3} \right\rfloor = 66 + 53 - 13 = 106$

- (b) (6) First write down 6 dec. length of inv. sequence = 6  
 (6, 5) Place 5 after  $i_5=1$  terms  
 (6, 4, 5) Place 4 after  $i_4=1$  terms  
 (6, 4, 5, 3) Place 3 after  $i_3=3$  terms  
 (6, 4, 2, 5, 3) Place 2 after  $i_2=2$  terms  
 (6, 4, 2, 5, 1, 3) Place 1 after  $i_1=4$  terms

2(a) (i) (Multinomial Theorem) For each  $n \in \mathbb{N}$ ,  $(x_1 + \dots + x_k)^n = \sum_{\substack{n_1 + \dots + n_k = n \\ n_i \in \mathbb{N}}} \binom{n}{n_1, \dots, n_k} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdots x_k^{n_k}$  where  $\binom{n}{n_1, \dots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$

a(ii) Coeff. of  $x^2yz^3$  in exp. of  $(3x-4y+2z)^6 = \binom{6}{2, 1, 3} \cdot (3)^2 \cdot (-4) \cdot (2)^3$   
 $= \frac{-6!}{2! 1! 3!} \cdot (3)(4)(8) = -3(720)(8) = -3(5760) = -17,280$ .

(b) Let  $X_1 = Y_1 + 1$ ,  $X_2 = Y_2 - 3$ , &  $X_3 = Y_3 + 4$ . Then number of integer solutions of  $X_1 + X_2 + X_3 = 10$  with  $X_1 \geq 1$ ,  $X_2 \geq -3$ , &  $X_3 \geq 4$   
 = no. of non-neg. integer solutions of  $(Y_1 + 1) + (Y_2 - 3) + (Y_3 + 4) = 10$   
 = no. of non-neg. integer solutions of  $Y_1 + Y_2 + Y_3 = 8$   
 = no. of permutations of  $[8, "1", 2, "+"] = \frac{10!}{8! 2!} = \frac{10 \cdot 9}{2} = 45$ .

3(a) No. of perm. of  $\{0, 1, \dots, 6\}$  with odds  $\rightarrow$  odds & each even  $\rightarrow$  itself  
 = (No. of perm. of  $\{1, 3, 5\}$ ). (No. of derangements of  $\{0, 2, 4, 6\}$ )  
 =  $(3!)(D_4) = 6(9) = 54$ .

3(b) Let  $U$  = set of all permutations of  $\{0, 1, 2, \dots, 6\}$  and  
 $A_i$  = set of perm. in  $U$  with  $i$  going to itself, for  $i=1, 3, 5$ .  
 $|U| = 7!$ ,  $|A_i| = 6!$ ,  $|A_i A_j| = 5!$ , and  $|A_i A_j A_k| = 4!$  for  
distinct  $i, j, k$ . So no. of permutations of  $\{0, 1, 2, \dots, 6\}$   
with no odd element going to itself =  $|A_1^c \cap A_3^c \cap A_5^c|$   
=  $|U| - |A_1| - |A_3| - |A_5| + |A_1 A_3| + |A_1 A_5| + |A_3 A_5| - |A_1 A_3 A_5|$   
=  $7! - 3(6!) + 3(5!) - 4!$

4(a) Answer = no. of perm. of  $[3.s, 1.u, 2.c, 1.e] = \frac{7!}{3!1!2!1!}$   
=  $(7.6.5.4)/2 = (21)(20) = 420$

(b) Let  $S = [\infty.a, \infty.b, \infty.c]$  &  $U$  = set of all 15-comb. of  $S$ .  
Put  $A$  = set of all 15-comb. of  $S$  with  $\geq 6a$ 's in each comb.  
= set of all 9-comb. of  $S$  with 6a's added to each comb,  
 $B$  = set of 15-comb. of  $S$  with  $\geq 8b$ 's in each comb.  
= set of 7-comb. of  $S$  with 8b's added to each comb.  
&  $C$  = set of 15-comb. of  $S$  with  $\geq 19c$ 's in each comb. =  $\emptyset$   
Then  $A \cap B$  = set of all 15-comb. of  $S$  with  $\geq 6a$ 's &  $8b$ 's in each.  
= set of all 1-comb. of  $S$  with 6a's & 8b's add to each.

$A \cap C = \emptyset$ ,  $B \cap C = \emptyset$ , and  $A \cap B \cap C = \emptyset$ . So number of  
15-comb. of  $M = [5.a, 7.b, 18.c] = |A^c \cap B^c \cap C^c|$

$$\begin{aligned} &= |U| - |A| - |B| - |C| + |AB| + |AC| + |BC| - |ABC| \\ &= \binom{15+3-1}{3-1} - \binom{9+3-1}{3-1} - \binom{7+3-1}{3-1} - 0 + \binom{1+3-1}{3-1} + 0 + 0 - 0 \\ &= \binom{17}{2} - \binom{11}{2} - \binom{9}{2} + \binom{3}{2} \end{aligned}$$

5(a)  $\binom{n}{k}$  = no. of  $k$ -subsets of  $\{1, 2, 3, \dots, n\}$   
= no. of  $k$ -subsets of  $\{1, 2, 3, \dots, n\}$  containing 1  
+ no. of  $k$ -subsets of  $\{1, 2, 3, \dots, n\}$  not containing 1  
=  $\binom{n-1}{k-1} + \binom{n-1}{k}$  choose  $k$  elements from  $\{2, 3, \dots, n\}$   
choose  $k-1$  elements from  $\{2, 3, \dots, n\}$  & add 1.

5(b) We will prove the result by parametric induction on  $n$  ( $k$  will be the temporarily fixed parameter).

Basis: For  $n=k$ , we have  $\binom{m}{k} = \binom{k}{k} = 1 = \binom{k+1}{k} = \binom{n+1}{k+1}$ . So the result is true for  $k$ .

Ind. Step: Suppose the result is true for  $n$ , where  $n \geq k$ . Then

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}. \text{ So}$$

$$\underbrace{\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k}}_{= \binom{n+1}{k}} + \binom{n+1}{k+1} = \binom{n+1}{k} + \binom{n+1}{k+1} = \binom{(n+1)+1}{k+1}$$

by Pascal's Identity.

So if the result is true for  $n$ , it will also be true for  $n+1$ . By the Principle of Parametric Mathematical Induction, it follows that the result is true for all  $n \geq k$ .

6(a) If  $k$  pigeons are placed in  $n$  holes, then there exists a hole with at least  $\lceil \frac{k}{n} \rceil$  pigeons for any  $k, n \in \mathbb{Z}^+$ .

$$\text{Note: } \lceil \frac{k-1}{n} \rceil = \lfloor \frac{k-1}{n} \rfloor + 1.$$

(b) Let  $S = \{a_1, a_1+a_2, a_1+a_2+a_3, \dots, a_1+a_2+\dots+a_n\}$ .

Then  $|S| = n$ . Let  $f: S \rightarrow \{0, 1, 2, \dots, n-1\}$  be defined by  $f(x) = x \pmod{n}$ . There are two cases.

Case(i): For some  $x \in S$ ,  $f(x) \equiv 0 \pmod{n}$ . In this case

$x = a_1+a_2+\dots+a_i$  for some  $i \in \mathbb{Z}^+$  and we get a non-empty segment whose terms add to  $0 \pmod{n}$ , i.e., whose terms add up to a multiple of  $n$ .

Case(ii): For each  $x \in S$ ,  $f(x) \not\equiv 0 \pmod{n}$ . In this case the only possibilities of  $f(x)$  are  $1, 2, 3, \dots, n-1$ . Since  $|S| = n$ , we can find two elements  $x_1 = a_1+a_2+\dots+a_i$  and

$x_2 = a_1+a_2+\dots+a_j$  with  $1 \leq i < j \leq n$  such that  $f(x_1) \equiv f(x_2) \pmod{n}$ .

So  $a_{i+1}+a_{i+2}+\dots+a_j \equiv 0 \pmod{n}$  & hence is a multiple of  $n$ .