

Answer all 6 questions. No Calculators or Cell phones are allowed. An unjustified answer will receive little credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

- (15) 1.(a) How many integers in the set $\{1, 2, 3, \dots, 1000\}$ are divisible by 10 or 12?
 (b) Find the permutation which has $\langle 3, 3, 1, 2, 0, 0 \rangle$ as its inversion sequence.
- (15) 2.(a) Write down what is the *Multinomial Theorem* in full detail. Then use it to find the coefficient of xyz^3 in the expansion of $(3x - 4y + z/2)^5$.
 (b) Find the number of integer solutions of the equation: $x_1 + x_2 + x_3 + x_4 = 10$ with $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, \text{ and } x_4 \geq -2$.
- (15) 3. (a) How many permutations of $\{1, 2, 3, \dots, 6\}$ have exactly 2 elements going to themselves?
 (a) How many permutations of $\{1, 2, 3, \dots, 6\}$ have no even elements going to itself? [You may leave your answer in terms of simplified factorials.]
- (20) 4.(a) Write down what is the *Inclusion-Exclusion Theorem* in full detail.
 (b) Find the number of 18-combinations of the multiset $F = [4.a, 10.b, 12.c]$. [You may leave your answer in terms of simplified Binomial coefficients.]
- (20) 5.(a) Using the Binomial Theorem, give an analytic proof of Pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$
 for all integers k, n with $1 \leq k \leq n-1$.
 (b) Hence, or otherwise, prove that for all integers $k, n \geq 0$ we have:

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \dots + \binom{n+k}{k} = \binom{n+k+1}{k}$$
.
- (15) 6.(a) Define what is a *positive set* with respect to the subsets A_1, A_2, \dots, A_n of the universal set U .
 (b) Using the Binomial Theorem, find the value of the sum:

$$\frac{1}{1}\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + \frac{1}{n+1}\binom{n}{n}$$
.

1(a) Let $U = \{1, 2, 3, \dots, 1000\}$, $A = \{x \in U : x \text{ is divisible by } 10\}$ and $B = \{x \in U : x \text{ is divisible by } 12\}$. Then $A \cap B = \{x \in U : x \text{ is divisible by } \text{lcm}(10, 12)\} = \{x \in U : x \text{ is divisible by } 60\}$. So answer
 $= |A \cup B| = |A| + |B| - |A \cap B| = \left\lfloor \frac{1000}{10} \right\rfloor + \left\lfloor \frac{1000}{12} \right\rfloor - \left\lfloor \frac{1000}{60} \right\rfloor = 100 + 83 - 16 = 167.$

(b) (6) First write down 6 b.c. length of seg = 6

(5, 6) Place 5 after 0 terms because $i_5 = 0$

(5, 6, 4) Place 4 after 2 terms because $i_4 = 2$

(5, 3, 6, 4) Place 3 after 1 term b.c. $i_3 = 1$

(5, 3, 6, 2, 4) Place 2 after 3 terms b.c. $i_2 = 3$

(5, 3, 6, 1, 2, 4) Place 1 after 3 terms b.c. $i_1 = 3$.

2(a) Multinomial Theorem: For each $k, n \in \mathbb{N}$, $(x_1 + \dots + x_k)^n = \sum_{n_1 + \dots + n_k = n, n_i \in \mathbb{N}} \binom{n}{n_1, \dots, n_k} \cdot x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$ where $\binom{n}{n_1, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$

$$\text{Coeff. of } xyz^3 \text{ in the expansion of } (3x - 4y + z/2)^5 = \binom{5}{1, 1, 3} \cdot (3) \cdot (-4) \cdot \binom{1}{2}^3 \\ = \frac{5!}{1! 1! 3!} \cdot \frac{(3) \cdot (-4)}{8} = \frac{5 \cdot 4}{1} \cdot \frac{3(-1)}{2} = -30.$$

(b) Let $x_1 = y_1 + 2$, $x_2 = y_2 + 3$, $x_3 = y_3 + 4$, $x_4 = y_4 - 2$. Then no. of integer solutions of $x_1 + x_2 + x_3 + x_4 = 10$ with $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 4$ & $x_4 \geq -2$
 = no. of non-neg. integer solutions of $(y_1 + 2) + (y_2 + 3) + (y_3 + 4) + (y_4 - 2) = 10$
 = no. of non-neg. integer solutions of $y_1 + y_2 + y_3 + y_4 = 3$
 = no. of permutations of $[3, "1", 3, "+"] = \frac{(3+3)!}{3! 3!} = \frac{6 \cdot 5 \cdot 4}{3!} = 20.$

3(a) No. of permutations of $\{1, 2, 3, \dots, 6\}$ with exactly 2 elements going to themselves = $(\text{No. of ways of sending 2 elements to themselves}) \cdot (\text{No. of ways of deranging the other four elements})$
 $= \binom{6}{2} \cdot D_4 = \frac{6 \cdot 5}{2 \cdot 1} \cdot 9 = 135 \text{ b.c. } D_4 = 4D_3 + (-1)^4 = 4(2) + 1 = 9.$

3(b) Let \mathcal{U} = set of all permutations of $\{1, 2, 3, \dots, 6\}$ and
 $A_{2i} = \{x \in \mathcal{U} : x_i \text{ goes to itself}\}$ for $i=1, 2, 3$. Then
 $|\mathcal{U}| = 6!$, $|A_2| = |A_4| = |A_6| = 5!$, $|A_2 A_4| = |A_2 A_6| = |A_4 A_6| = 4!$
and $|A_2 A_4 A_6| = 3!$ So no. of permutations of $\{1, 2, 3, \dots, 6\}$
with no even integer going to itself $= |A_2^c A_4^c A_6^c| =$
 $|\mathcal{U}| - (|A_2| - |A_4| - |A_6| + |A_2 A_4| + |A_2 A_6| + |A_4 A_6| - |A_2 A_4 A_6|)$
 $= 6! - 3 \cdot 5! + 3 \cdot 4! - 3!$

4(a) Inclusion-Exclusion Theorem : Let A_1, \dots, A_n be subsets of a universal set \mathcal{U} . Then $|A_1^c A_2^c \dots A_n^c| =$
 $\sum_{k=0}^n (-1)^k \left\{ \sum_{\substack{(i_1, \dots, i_k) \in IS(k,n)}} |A_{i_1} A_{i_2} \dots A_{i_k}| \right\}$ where $IS(k,n)$ is the set of all increasing subsequences $\langle i_1, \dots, i_k \rangle$ of $\langle 1, 2, \dots, n \rangle$. When $k=0$, $\langle i_1, \dots, i_k \rangle = \langle \rangle$, the empty subsequence.

(b) Let $S = [\infty, a, \infty, b, \infty, c]$ & \mathcal{U} = set of all 18-comb. of S . Put
 A = set of all 18-comb. of S with $\geq 5a$'s
= set of all 13-comb. of S with 5a's added to each 13-comb.
 B = set of all 18-comb. of S with $\geq 11b$'s
= set of all 7-comb. of S with 11b's added to each 7-comb,
& C = set of all 18-comb. of S with $\geq 13c$'s
= set of all 5-comb. of S with 13b's added to each 5-comb,

Then AB = set of all 2-comb. of S with 5a's & 11b's added,
 AC = set of all 0-comb. of S with 5a's & 13c's added,
 $BC = \emptyset$, and $ABC = \emptyset$. So, by the Inclusion-Exclusion Thm,
no. of 18-comb. of $F = [4.a, 10.b, 12.c]$, is $|A^c B^c C^c|$
 $= |\mathcal{U}| - |A| - |B| - |C| + |AB| + |AC| + |BC| - |ABC|$
 $= \binom{18+3-1}{3-1} - \binom{13+3-1}{3-1} - \binom{7+3-1}{3-1} - \binom{5+3-1}{3-1} + \binom{2+3-1}{3-1} + \binom{0+3-1}{3-1} + 0 - 0$
 $= \binom{20}{2} - \binom{15}{2} - \binom{9}{2} - \binom{7}{2} + \binom{4}{2} + \binom{2}{2}$.

$$5(a) \text{ Binomial Theorem: } (1+x)^n = \binom{n}{0}x^0 + \dots + \binom{n}{k}x^k + \dots + \binom{n}{n}x^n. \text{ So } (1+x)^n =$$

$$(1+x)(1+x)^{n-1} = (1+x) \left[\binom{n-1}{0}x^0 + \dots + \binom{n-1}{k-1}x^{k-1} + \binom{n-1}{k}x^k + \dots + \binom{n-1}{n-1}x^{n-1} \right]$$

$$= 1 \binom{n-1}{0} \cdot x^0 + \dots + \left\{ \binom{n-1}{k-1}x^{k-1}, x + 1, \binom{n-1}{k}x^k + \dots + \binom{n-1}{n-1}x^{n-1} \right\} x$$

Comparing the coefficients of x^k in both equations gives us

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

(b) We will prove the result by parametric induction on k . (n will be the fixed parameter). Since $\binom{n}{0} = 1 = \binom{n+0+1}{0}$, the result is clearly true for $k=0$. Now suppose the result is true for k . Then $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \dots + \binom{n+k}{k} = \binom{n+k+1}{k}$. So $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \dots + \binom{n+k}{k} + \binom{n+k+1}{k+1} = \binom{n+k+1}{k} + \binom{n+k+1}{k+1}$ $= \binom{n+(k+1)+1}{k+1}$ by Pascal's identity. So if the result is true for k , it will be true for $k+1$. By the Principle of Parametric Math. Induction, it follows that the result is true for all k, n .

6(a) Let A_1, A_2, \dots, A_n be subsets of a universal set U . A positive sets w.r.t. U & A_1, \dots, A_n is any set of the form $U \cap A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}$ where $\langle i_1, i_2, \dots, i_k \rangle$ is an increasing subsequence of $\langle 1, 2, \dots, n \rangle$. When $k=0$, $\langle i_1, \dots, i_k \rangle = \langle \rangle$, the empty subsequence of $\langle 1, 2, \dots, n \rangle$ which is increasing.

(b) From the Binomial Theorem we know that

$$\binom{n}{0} \cdot x^0 + \binom{n}{1} \cdot x^1 + \binom{n}{2} \cdot x^2 + \dots + \binom{n}{n} \cdot x^n = (1+x)^n. \text{ So}$$

$$\int_0^1 \left[\binom{n}{0} \cdot x^0 + \binom{n}{1} \cdot x^1 + \binom{n}{2} \cdot x^2 + \dots + \binom{n}{n} \cdot x^n \right] dx = \int_0^1 (1+x)^n dx.$$

$$\therefore \left[\frac{1}{1} \binom{n}{0} x^1 + \frac{1}{2} \binom{n}{1} x^2 + \frac{1}{3} \binom{n}{2} x^3 + \dots + \frac{1}{n+1} \binom{n}{n} x^{n+1} \right]_0^1 = \left[\frac{(1+x)^{n+1}}{n+1} \right]_0^1$$

$$\therefore \frac{1}{1} \binom{n}{0} + \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} + \dots + \frac{1}{n+1} \binom{n}{n} = \frac{2^{n+1} - 1}{n+1}.$$

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