

Answer all 6 questions. No Calculators or Cellphones are allowed. An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

- (15) 1. Find the solution of the recurrence equation $x_{n+2} - x_{n+1} - 6x_n = 0$ with the initial conditions $x_0 = 3$ and $x_1 = 4$.
- (20) 2. Find the general solution of the following recurrence equations:
(a) $x_{n+2} - 4x_n = 6n$, (b) $x_{n+2} + 2x_{n+1} - 3x_n = 6$.
- (20) 3(a) Use the method of generating functions to find the solution of the recurrence equation: $a_n - 3a_{n-1} + 2 = 0$ with $a_0 = 3$.
(b) Let $h_k = 3k^2 + 3k + 1$. Find the zero-column of $\langle h_k \rangle_{k \in \mathbb{N}}$ and a formula for the sum $h_0 + h_1 + h_2 + \dots + h_n$. [Simplify your answer as far as possible.]
- (15) 4. Prove that in any group of 10 people, we can always find 3 mutual acquaintances or 4 mutual strangers. (Note: stranger = non-acquaintance). [You may use the fact that in any group of 6 people, we can always find 3 mutual acquaintances or 3 mutual strangers, if needed in Problem #4.]
- (15) 5(a) Define what are the Stirling numbers of the First kind.
(b) Prove for any k with $0 < k < p$, the Stirling numbers of the first kind satisfy,
$$\begin{bmatrix} p \\ k \end{bmatrix} = \begin{bmatrix} p-1 \\ k-1 \end{bmatrix} + (p-1) \cdot \begin{bmatrix} p-1 \\ k \end{bmatrix}.$$
- (15) 6(a) Let $p, k \in \mathbb{N}$. Define what is the combinatorial expression $S(p, k)$.
(b) Give a combinatorial proof that for any $p, k \in \mathbb{N}$ with $0 < k < p$, we have
$$S(p, k) = S(p-1, k-1) + k \cdot S(p-1, k).$$

1. $X_{n+2} - X_{n+1} - 6X_n = 0$, $\therefore (E^2 - E - 6I)X_n = 0$. Hence $(E-3I)(E+2I)X_n = 0$. $\therefore X_n = A.(3)^n + B.(-2)^n$ is the general solution. Now

$$X_0 = 3 \Rightarrow A(3)^0 + B(-2)^0 = 3 \Rightarrow A+B=3 \quad \text{and}$$

$$X_1 = 4 \Rightarrow A(3)^1 + B(-2)^1 = 4 \Rightarrow 3A-2B=4.$$

$$\therefore B = 3-A, \text{ so } 3A-2(3-A) = 4 \Rightarrow 5A-6=4 \Rightarrow A=2$$

$$\text{Also } B = 3-2=1. \quad \therefore X_n = 2.(3)^n + 1.(-2)^n.$$

2 (a) $X_{n+2} - 4X_n = 6n$. $\therefore (E^2 - 4I)X_n = 6n$. So $(E-I)^2(E^2-4I)X_n = (E-I)^2(6n)$. $\therefore (E-I)^2(E-2I)(E+2I)X_n = (E^2-2E+I)(6n) = 6(n+2) - 2\cdot 6(n+1) + 6n = 6n - 12n + 6n + 12 - 12 = 0$.

$$\therefore (E-I)^2(E-2I)(E+2I)X_n = 0. \quad \text{Hence}$$

$$X_n = A.(2)^n + B.(-2)^n + (Cn+D).(1)^n. \quad \text{But } X_{n+2} - 4X_n = 6n$$

$$\therefore A(2)^{n+2} + B(-2)^{n+2} + [C(n+2)+D] - 4[A(2)^n + B(-2)^n + 4(Cn+D)] = 6n$$

$$\therefore (C-4C)n + (2C+D-4D) = 6n + 0.$$

$$\therefore -3C = 6 \quad \text{and} \quad 2C-3D = 0.$$

$$\therefore C = -2 \quad \text{and} \quad D = 2C/3 = -4/3.$$

$$\text{Hence } X_n = A.(2)^n + B.(-2)^n - 2n - 4/3.$$

(b) $X_{n+2} + 2X_{n+1} - 3X_n = 6 \Rightarrow (E^2 + 2E - 3I)X_n = 6$. Hence

$$(E-I)[(E^2 + 2E - 3I)X_n] = (E-I)6 = 6-6 = 0$$

$$\therefore (E-I)(E-I)(E+3)X_n = 0 \Rightarrow (E-I)^2(E+3) = 0.$$

$$\therefore X_n = A.(-3)^n + (B+Cn).(1)^n. \quad \text{But } X_{n+2} + 2X_{n+1} - 3X_n = 6$$

$$\therefore A(-3)^{n+2} + [B+C(n+2)] + 2.A(-3)^{n+1} + 2.[B+C(n+1)] - 3A(-3)^n - 3(B+Cn) = 6$$

$$\therefore (9A-6A-3A)(-3)^n + (B+2B-3B) + (2C+2C) + (C+2C-3C)n = 6$$

$$\therefore 4C = 6. \quad \text{Hence } C = 6/4 = 3/2. \quad \text{Thus}$$

$$X_n = A.(-3)^n + \left(B + \frac{3n}{2}\right)(1)^n = A.(-3)^n + B + 3n/2.$$

3(a) Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$
 Then $-3x f(x) = -3a_0 x - 3a_1 x^2 - \dots - 3a_{n-1} x^n + \dots$
 and $2/(1-x) = 2 + 2x + 2x^2 + \dots + 2x^n + \dots$

$$\begin{aligned}\therefore (1-3x)f(x) + \frac{2}{1-x} &= (a_0 + 2) + (a_1 - 3a_0 + 2)x + \dots + (a_n - 3a_{n-1} + 2)x^n + \dots \\ &= (3+2) + 0.x + 0.x^2 + \dots + 0.x^n + \dots = 5\end{aligned}$$

$$(1-3x)f(x) = 5 - 2/(1-x) = [5(1-x) - 2]/(1-x) = (3-5x)/(1-x)$$

$$\therefore f(x) = \frac{3-5x}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

$$\therefore 3-5x = A(1-x) + B(1-3x).$$

$$\text{Putting } x=1/3, \text{ gives us } 3-5/3 = A(1-1/3) \Rightarrow A=2$$

$$\text{Putting } x=1, \text{ gives us } 3-5(1) = B(1-3(1)) \Rightarrow B=1$$

$$\begin{aligned}\therefore f(x) &= \frac{2}{1-3x} + \frac{1}{1-x} = 2[1 + (3x) + (3x)^2 + \dots + (3x)^n + \dots] \\ &\quad + 1[1 + x + x^2 + \dots + x^n + \dots]\end{aligned}$$

$$\therefore a_n = \text{coeff. of } x^n \text{ in this exp.} = 2 \cdot (3)^n + 1.$$

$$(b) h_k = 1 \quad 7 \quad 19 \quad 37 \quad 61 \quad \dots$$

$$\langle \Delta h_k \rangle = 6 \quad 12 \quad 18 \quad 24 \quad \dots$$

$$\langle \Delta^2 h_k \rangle = 6 \quad 6 \quad 6 \quad \dots$$

$$\langle \Delta^3 h_k \rangle = 0 \quad 0 \quad \dots \quad \therefore \text{zero column} = \langle 1, 6, 6, 0, \dots \rangle$$

$$\begin{aligned}\therefore h_k &= 1 \cdot \binom{k}{0} + 6 \cdot \binom{k}{1} + 6 \cdot \binom{k}{2}. \quad \therefore \sum_{k=0}^n h_k = \sum_{k=0}^n \left\{ 1 \cdot \binom{k}{0} + 6 \cdot \binom{k}{1} + 6 \cdot \binom{k}{2} \right\} \\ &= 1 \cdot \binom{n+1}{1} + 6 \cdot \binom{n+1}{2} + 6 \cdot \binom{n+1}{3} = \frac{(n+1)}{1!} + \frac{6(n+1)n}{2!} + \frac{6(n+1)n(n-1)}{3!} = (n+1)^3.\end{aligned}$$

4. Choose one person and call her p_i - and let $A = \text{set of all acquaintances of } p_i$ & $S = \text{set of all strangers to } p_i$. Then $|A \cup S| = 9$ and $A \cap S = \emptyset$. So $|A| + |S| = 9$. Hence $|A| \geq 4$ or $|S| \geq 6$. We have 2 cases.

Case(i): $|A| \geq 4$. In this case either all the people in A are mut. strangers or A contains 2 acq. Now if all the people in A are mut. strangers, we get 4 mut. strangers. And if A contains 2 acq. we just add p_i to get 3 mut. acquaintances.

Case(ii) $|S| \geq 6$. In this case either S contains 3 mut. strangers or S contains 3 mut. friends. If S contains 3 mut. strangers we just add p_i to get 4 mut. str. And if S contains 3 mut. acq, we

4. We get 3 mut. acq. So in both cases we got what was needed

5(a) The Stirling numbers of the First kind are the unique integers $\begin{Bmatrix} p \\ k \end{Bmatrix}$ such that $[n]_p = n(n-1)(n-2)\dots(n-(p-1)) = \sum_{k=0}^p (-1)^{p-k} \begin{Bmatrix} p \\ k \end{Bmatrix} \cdot n^k$. (*)

$$\begin{aligned}
 (b) [n]_p &= [n]_{p-1} \cdot [n - (p-1)] = \sum_{k=0}^{p-1} (-1)^{p-1-k} \begin{Bmatrix} p-1 \\ k \end{Bmatrix} n^k \cdot [n - (p-1)] \\
 &= \sum_{k=0}^{p-1} (-1)^{p-1-k} \begin{Bmatrix} p-1 \\ k \end{Bmatrix} n^{k+1} - \sum_{k=0}^{p-1} (-1)^{p-1-k} \begin{Bmatrix} p-1 \\ k \end{Bmatrix} \cdot (p-1) \cdot n^k \\
 &= \sum_{k=1}^p (-1)^{p-k} \begin{Bmatrix} p-1 \\ k-1 \end{Bmatrix} n^k + \sum_{k=0}^{p-1} (-1)^{p-k} \cdot (p-1) \begin{Bmatrix} p-1 \\ k \end{Bmatrix} \cdot n^k \\
 &\quad [\text{replace } k \text{ by } k-1] \quad [\text{put } "-" \text{ sign inside the } \sum] \\
 &= \begin{Bmatrix} p-1 \\ p-1 \end{Bmatrix} n^p + \sum_{k=1}^{p-1} \left\{ \begin{Bmatrix} p-1 \\ k-1 \end{Bmatrix} n^k + (p-1) \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix} n^k \right\} + (-1)^p (p-1) \begin{Bmatrix} p-1 \\ 0 \end{Bmatrix} n^0
 \end{aligned}$$

Comparing the coefficients of $[n]_p$ from (*) and this equation we get for $1 \leq k \leq p$, $\begin{Bmatrix} p \\ k \end{Bmatrix} = \begin{Bmatrix} p-1 \\ k-1 \end{Bmatrix} + (p-1) \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix}$.

6(a) $S(p, k)$ is the number of partitions of $\{1, 2, 3, \dots, p\}$ into k non-empty parts.

(b) Let A = set of all partitions of $\{1, 2, \dots, p\}$ into k non-empty parts, B = set of partitions in A in which p is in a part by itself, and B' = set of partitions in A in which p is not in a part by itself. Then $A = B \cup B'$ and $B \cap B' = \emptyset$. So $|A| = |B| + |B'|$.

Now if we take any partition of B and we remove the part $\{p\}$, we will get a partition of $\{1, 2, \dots, p-1\}$ into $k-1$ non-empty parts. And if we add $\{p\}$ as a new part to any partition of $\{1, 2, \dots, p-1\}$ into $k-1$ non-empty parts, we will get a partition of B . $\therefore |B| = S(p-1, k-1)$.

Also if we take any partition of B' and we remove p from the part in which p is contained, we will get a partition of $\{1, 2, \dots, p-1\}$ into k non-empty parts. And if we add p (in turns) to each of the k parts of a partition of $\{1, 2, \dots, p-1\}$ into k non-empty parts, we will k partitions of B' . Hence $|B'| = k \cdot S(p-1, k)$. $\therefore S(p, k) = |A| = |B| + |B'| = S(p-1, k-1) + k \cdot S(p-1, k)$.