

*Answer all 6 questions. No Calculators or Cell phones are allowed. An unjustified answer will receive little credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.*

- (16) 1.(a) How many *integers* in the set  $\{1, 2, 3, \dots, 700\}$  are *divisible* by 9 or 12?  
(b) Find the *permutation* which has  $\langle 4, 2, 3, 1, 1, 0 \rangle$  as its *inversion sequence*.
- (16) 2.(a) Write down what is the *Multinomial Theorem* in full detail. Then use it to find the *coefficient of  $x^3y^2z$*  in the expansion of  $(2x - 3y + z/4)^6$ .  
(b) Find the number of *integer solutions* of the equation:  $x_1 + x_2 + x_3 = 12$  with  $x_1 \geq 3$ ,  $x_2 \geq -4$ , and  $x_3 \geq 5$ .
- (16) 3.(a) Write down what is the *Inclusion-Exclusion Theorem* in full details.  
(b) Find the number of *16-combinations* of the multi-set  $F = [5.a, 8.b, 14.c]$ .  
[You may leave your answer in terms of simplified Binomial coefficients.]
- (16) 4.(a) Define what is a *positive set* with respect to the subsets  $A_1, A_2, \dots, A_n$  of the universal set  $U$ .  
(b) Using the Binomial Theorem, find the value of the sum:  
$$2^0 \cdot \frac{1}{1} \binom{n}{0} + 2^1 \cdot \frac{1}{2} \binom{n}{1} + 2^2 \cdot \frac{1}{3} \binom{n}{2} + 2^3 \cdot \frac{1}{4} \binom{n}{3} + \dots + 2^n \cdot \frac{1}{n+1} \binom{n}{n}.$$
- (18) 5. (a) How many permutations of  $\{1, 2, 3, 4, 5, 6, 7\}$  have *each odd number* going to an odd number, and *no even number* going to itself  
(b) How many permutations of  $\{1, 2, 3, 4, 5, 6, 7\}$  have *no even number* going to itself? [You may leave your answer in terms of simplified factorials.]
- (18) 6.(a) Give a *combinatorial proof* of Pascal's identity:  
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$
 for all integers  $k$  and  $n$  with  $1 \leq k \leq n-1$ .  
(b) Let  $B_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \dots + \binom{1}{n-1} + \binom{0}{n}$ .  
Using part (a), or otherwise, prove that  $B_{n-1} + B_{n-2} = B_n$  for  $n \geq 2$ .

## Solutions to Test #1

Fall 2016

1(a) Let  $U = \{1, 2, 3, \dots, 700\}$ ,  $A = \{x \in U : x \text{ is divisible by } 9\}$  &  $B = \{x \in U : x \text{ is divisible by } 12\}$ . Then  $A \cap B = \{x \in U : x \text{ is divisible by } \text{lcm}(9, 12)\} = \{x \in U : x \text{ is divisible by } 36\}$ . So answer =  $|A \cup B| = |A| + |B| - |A \cap B| = \left\lfloor \frac{700}{9} \right\rfloor + \left\lfloor \frac{700}{12} \right\rfloor - \left\lfloor \frac{700}{36} \right\rfloor$   
 $= \left\lfloor \frac{77}{9} \right\rfloor + \left\lfloor \frac{175}{3} \right\rfloor - \left\lfloor \frac{175}{9} \right\rfloor = 77 + 58 - 19 = 116$ .

- 1(b) <6> First write down 6 b.c. length of inv. seg. = 6  
<6,5> Place 5 after 1 term because  $i_5 = 1$   
<6,4,5> Place 4 after 1 term because  $i_4 = 1$   
<6,4,5,3> Place 3 after 3 terms b.c.  $i_3 = 3$   
<6,4,2,5,3> Place 2 after 2 terms b.c.  $i_2 = 2$   
<6,4,2,5,1,3> Place 1 after 4 terms b.c.  $i_1 = 4$

2(a) Multi-nomial Theorem: For each  $k, n \in \mathbb{N}$ ,  $(x_1 + x_2 + \dots + x_k)^n = \sum_{\{(n_1, \dots, n_k) : n_1 + \dots + n_k = n\}} \binom{n}{n_1, \dots, n_k} \cdot x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$  where  $\binom{n}{n_1, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$

Coeff. of  $x^3 y^2 z$  in the expansion of  $(2x - 3y + z/4)^6$   
 $= \binom{6}{3, 2, 1} \cdot (2)^3 \cdot (-3)^2 \cdot (1/4)^1 = \frac{6!}{3! 2! 1!} \cdot 2^3 \cdot 9/4^2 = \frac{6 \cdot 5 \cdot 4}{2} \cdot 2(9) = 1080$ .

- 2(b) Let  $x_1 = y_1 + 3$ ,  $x_2 = y_2 - 4$ ,  $x_3 = y_3 + 5$ . Then no. of integer sol. of  $x_1 + x_2 + x_3 = 12$  with  $x_1 \geq 3$ ,  $x_2 \geq -4$ , and  $x_3 \geq 5$   
= no. of non-neg. integer solutions of  $(y_1 + 3) + (y_2 - 4) + (y_3 + 5) = 12$   
= no. of non-neg. integer sol. of " $y_1 + y_2 + y_3 = 8$ " =  $\binom{8+3-1}{3-1} = 45$ .

3(a) Inclusion-Exclusion Theorem: For any subsets  $A_1, \dots, A_n$  of  $U$   
 $|A_1^c A_2^c \dots A_n^c| = \sum_{k=0}^n (-1)^k \left\{ \sum_{\{(i_1, \dots, i_k) \in IS(k, n)\}} |\cup A_{i_1} A_{i_2} \dots A_{i_k}| \right\}$  where

$IS(k, n)$  = set of all subsequences of  $\langle 1, 2, \dots, n \rangle$  of length  $k$ .

3(b) Let  $S = [\infty, a, \infty, b, \infty, c]$  &  $\mathcal{U}$  = set of all 16-comb. of  $S$ . Put

$A$  = set of all 16-combinations of  $S$  with  $\geq 6a$ 's

= set of all 10-comb. of  $S$  with 6a's added to each 10-comb.

$B$  = set of all 16-combinations of  $S$  with  $\geq 9b$ 's

= set of all 7-comb. of  $S$  with 9b's added to each 7-comb.

&  $C$  = set of all 16-combinations of  $S$  with  $\geq 15c$ 's

= set of all 1-comb. of  $S$  with 15c's added to each 1-comb.

Then  $AB$  = set of 1-comb. of  $S$  with 6a's & 9b's add to each 1-comb.

and  $AC = \emptyset$ ,  $BC = \emptyset$ , and  $ABC = \emptyset$ . So by the Incl.-Excl.

Theorem, no. of 16-comb. of  $F = [5.a, 8.b, 14.c] = |A^c B^c C^c|$

$$= |\mathcal{U}| - |A| - |B| + |AB| + |AC| + |BC| - |ABC|$$

$$= \binom{16+3-1}{3-1} - \binom{10+3-1}{3-1} - \binom{7+3-1}{3-1} - \binom{1+3-1}{3-1} + \binom{1+3-1}{3-1} + 0 + 0 - 0$$

$$= \binom{18}{2} - \binom{12}{2} - \binom{9}{2} - \binom{3}{2} + \binom{3}{2} = \binom{18}{2} - \binom{12}{2} - \binom{9}{2}.$$

4(a) A positive set w.r.t.  $\mathcal{U}$  and  $A_1, \dots, A_n$  is any expression of the form  $\cup_{i_1} A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}$  where  $\langle i_1, \dots, i_k \rangle$  is a subsequence of  $\langle 1, 2, 3, \dots, n \rangle$  with  $0 \leq k \leq n$ .

4(b) From the Standard Binomial Theorem we have

$$\binom{n}{0} x^0 + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \dots + \binom{n}{n} x^n = (1+x)^n$$

$$\therefore \int_0^2 \left[ \binom{n}{0} x^0 + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \dots + \binom{n}{n} x^n \right] dx = \int_0^2 (1+x)^n dx.$$

$$\therefore \left[ \binom{n}{0} \frac{x^1}{1} + \binom{n}{1} \frac{x^2}{2} + \binom{n}{2} \frac{x^3}{3} + \dots + \binom{n}{n} \frac{x^{n+1}}{n+1} \right]_0^n = \left[ \frac{(1+x)^{n+1}}{n+1} \right]_0^n$$

$$\therefore 2^1 \cdot \frac{1}{1} \binom{n}{0} + 2^2 \cdot \frac{1}{2} \binom{n}{1} + 2^3 \cdot \frac{1}{3} \binom{n}{2} + \dots + 2^{n+1} \cdot \frac{1}{n+1} \binom{n}{n} = \frac{3^{n+1} - 1}{n+1}^{n+1}$$

$$\therefore 2^0 \cdot \frac{1}{1} \binom{n}{0} + 2^1 \cdot \frac{1}{2} \binom{n}{1} + 2^2 \cdot \frac{1}{3} \binom{n}{2} + \dots + 2^n \cdot \frac{1}{n+1} \binom{n}{n} = \frac{1}{2} \cdot \frac{(3^{n+1} - 1)}{n+1}.$$

5(a) No. of perm. of  $\{1, 2, 3, 4, 5, 6, 7\}$  with each odd no. going to an odd no. & no even no. going to itself

$$= (\text{no. of perm. of } \{1, 3, 5, 7\}) \cdot (\text{no. of derangements of the set } \{2, 4, 6\}) = 4! D_3 = 24(2) = 48$$

5(b) Let  $\mathcal{U}$  = set of all permutations of  $\{1, 2, 3, \dots, 7\}$ . Put  
 $A_{2i} = \{x \in \mathcal{U} : 2i \text{ goes to itself}\}$  for  $i=1, 2, 3$ . Then  
 $|\mathcal{U}| = 7!$ ,  $|A_2| = |A_4| = |A_6| = 6!$ ,  $|A_2 A_4| = |A_2 A_6| = |A_4 A_6| = 5!$ ,  
and  $|A_2 A_4 A_6| = 4!$  So no. of perm. of  $\{1, 2, \dots, 7\}$  with  
no even no. going to itself  $= |A_2^c A_4^c A_6^c|$   
 $= |\mathcal{U}| - |A_2| - |A_4| - |A_6| + |A_2 A_4| + |A_2 A_6| + |A_4 A_6| - |A_2 A_4 A_6|$   
 $= 7! - 3(6!) + 3(5!) - 4! = (7-3)(6!) + [3(5)-1](4!)$   
 $= (2 \cdot 6 \cdot 5 \cdot 7) \cdot 2 \cdot (4!) = (53)(2)(4!) = (106)(4!).$

6(a)  $\binom{n}{k}$  = Number of  $k$ -subsets of  $\{1, 2, 3, \dots, n-1, n\}$   
 $= |\text{set of } k\text{-subsets of } \{1, 2, 3, \dots, n\} \text{ containing } n|$   
 $+ |\text{set of } k\text{-subsets of } \{1, 2, \dots, n\} \text{ not containing } n|$   
 $= |\text{set of } (k-1)\text{-subsets of } \{1, 2, 3, \dots, n-1\} \text{ with } n \text{ added to each subset}|$   
 $+ |\text{set of } k\text{-subsets of } \{1, 2, 3, \dots, n-1\}|$   
 $= \binom{n-1}{k-1} + \binom{n-1}{k}.$  (Pascal's identity)

(b)  $B_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \dots + \binom{1}{n-1} + \binom{0}{n}$ . So

$$\begin{aligned} B_{n+2} + B_{n+1} &= \binom{n+2}{0} + \binom{n+1}{1} + \binom{n}{2} + \dots + \binom{1}{n-1} + \binom{0}{n} \\ &\quad + \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \dots + \binom{1}{n-2} + \binom{0}{n-1} \\ &= \underbrace{\binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \dots + \binom{n+1}{n-2} + \binom{n+1}{n-1}}_{\text{all by Pascal's identity}} + 0. \end{aligned}$$

$$= \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \dots + \binom{2}{n-2} + \binom{1}{n-1} + \binom{0}{n} = B_n$$

because  $\binom{n-1}{0} = 1 = \binom{n}{0}$  &  $0 = \binom{0}{n}$  because  $n \geq 2$ .