

Answer all 6 questions. **No Calculators or Cellphones are allowed.** An **unjustified** answer will receive little or no credit. **BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.**

- (15) 1. Find *the solution* of the recurrence equation $x_{n+2} - 3x_{n+1} - 10x_n = 0$ with the *initial conditions* $x_0 = 5$ and $x_1 = 4$.
- (20) 2(a) Use the *method of generating functions* to find the solution of the recurrence equation: $a_n + 2a_{n-1} - 6 = 0$ with $a_0 = 5$.
- (b) Let $M = [\infty.a, 5.b, \infty.c]$. Find the *generating function* for the set of all n -combinations of M with *an even number of a's, an odd number of b's, and at least three c's.* [Simplify your answer as far as possible.]
- (15) 3(a) Define what is the *zero-column* (zero-diagonal) of the sequence $\langle h_k \rangle_{k \in \mathbb{N}}$.
- (b) Let $h_k = 3k^2 - 5k + 1$. Find the zero-column of $\langle h_k \rangle_{k \in \mathbb{N}}$ and a formula for the sum $h_0 + h_1 + h_2 + \dots + h_n$. [Simplify your answer as far as possible.]
- (15) 4(a) Define what are the *Stirling coefficients* of the *1st kind* and of the *2nd kind*.
- (b) Prove for any k with $0 < k < p$, the Stirling coefficients of the 2nd kind satisfy,
- $$\begin{Bmatrix} p \\ k \end{Bmatrix} = \begin{Bmatrix} p-1 \\ k-1 \end{Bmatrix} + k \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix}.$$
- (15) 5. Prove that in any group of 20 people, we can always find 4 mutual acquaintances or 4 mutual strangers. (Note: stranger = non-acquaintance). [You may use the fact that in any group of 10 people, we can always find 4 mutual acquaintances or 3 mutual strangers; and can also find 3 mutual acquaintances or 4 mutual strangers – if needed.]
- (20) 6. Find *the general solution* of the following recurrence equations:
- (a) $x_{n+2} + 2x_{n+1} + 2x_n = 10n$, (b) $x_{n+2} - x_{n+1} - 2x_n = 6 \cdot 2^n$.

i. $X_{n+2} - 3X_{n+1} - 10X_n = 0 \quad \therefore (E^2 - 3E - 10I)X_n = 0.$

Hence $(E+2I)(E-5I)X_n = 0.$

$\therefore X_n = A(-2)^n + B(5)^n$ is the general solution.

$X_0 = 5 \Rightarrow 5 = A(-2)^0 + B(5)^0 \Rightarrow A + B = 5 \Rightarrow B = 5 - A$

$X_1 = 4 \Rightarrow 4 = A(-2)^1 + B(5)^1 \Rightarrow -2A + 5B = 4.$

$\therefore -2A + 5(5 - A) = 4 \Rightarrow -7A = -21 \Rightarrow A = 3. \text{ So } B = 5 - 3 = 2.$

$\therefore X_n = A(-2)^n + B(5)^n = 3(-2)^n + 2(5)^n.$

2(a) Let $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$

Then $2xf(x) = 2a_0x + 2a_1x^2 + \dots + 2a_{n-1}x^n + \dots$

and $\frac{-6}{1-x} = -6 - 6x - 6x^2 - \dots - 6x^n - \dots$

$\therefore (1+2x)f(x) - \frac{6}{1-x} = a_0 - 6 + 0x + 0x^2 + \dots + 0x^n = 5 - 6 = -1$

$\therefore (1+2x)f(x) = \frac{6}{1-x} - 1 = \frac{6}{1-x} - \frac{1-x}{1-x} = \frac{5+x}{1-x}$

$\therefore f(x) = \frac{5+x}{(1+2x)(1-x)} = \frac{A}{1-x} + \frac{B}{1+2x}, \quad \therefore 5+x = A(1+2x) + B(1-x)$

Putting $x=1$, gives us $5+1 = A(1+2) \Rightarrow 3A=6 \Rightarrow A=2$

Putting $x=-\frac{1}{2}$ gives us $5-\frac{1}{2} = B(1-\frac{1}{2}) \Rightarrow \frac{3}{2}B = \frac{9}{2} \Rightarrow B=3$

$\therefore f(x) = \frac{2}{1-x} + \frac{3}{1+2x} = 2[1+x+x^2+\dots+x^n+\dots] + 3[1+(-2x)+(-2x)^2+\dots+(-2x)^n+\dots]$

$\therefore a_n = \text{coeff. of } x^n = 2 \cdot (1)^n + 3 \cdot (-2)^n = 2 + 3(-2)^n$

(b) Generating function for the set of all n -comb. of $\{\infty a, 5 b, \infty c\}$

with even no. of a 's, odd nos. of b 's, and at least 3 c 's is

$(x^0 + x^2 + x^4 + \dots)(x^1 + x^3 + x^5)(x^3 + x^4 + x^5 + \dots)$

$= [1 + x^2 + (x^2)^2 + \dots] x \cdot [1 + x^2 + (x^2)^2] \cdot x^3 [1 + x + x^2 + \dots]$

$= x^4 (1-x^2)^{-1} \cdot (1+x^2+x^4) \cdot (1-x)^{-1} = \frac{x^4 (1+x^2+x^4)}{(1-x^2)(1-x)}$

$= \frac{x^4 (1+x^2+x^4)}{(1-x) \cdot (1+x)(1-x)} = \frac{x^4 (1+x^2+x^4)}{(1-x)^2 (1+x)}$

3(a) The zero-column of $\langle h_n \rangle_{n \in \mathbb{N}}$ is $\langle \Delta^k h_0 \rangle_{k \in \mathbb{N}}$ where $\Delta^0 h_n = h_n$, $\Delta^{k+1} h_n = \Delta(\Delta^k h_n)$ and $\Delta h_n = h_{n+1} - h_n$ for $n \in \mathbb{N}$.

(b) $h_k = 3k^2 - 5k + 1$.

k	0	1	2	3	4	5
$\Delta^0 h_k$	1	-1	3	13	29	51
$\Delta^1 h_k$	-2	4	10	16	22	
$\Delta^2 h_k$	6	6	6	6		
$\Delta^3 h_k$	0	0	0			

\therefore Zero-column of h_k is $\langle 1, -2, 6, 0, 0, \dots \rangle$

$\therefore h_k = 1 \cdot \binom{k}{0} - 2 \cdot \binom{k}{1} + 6 \cdot \binom{k}{2}$ and hence

$$\begin{aligned} \sum_{k=0}^n h_k &= 1 \cdot \sum_{k=0}^n \binom{k}{0} - 2 \cdot \sum_{k=0}^n \binom{k}{1} + 6 \cdot \sum_{k=0}^n \binom{k}{2} = 1 \cdot \binom{n+1}{1} - 2 \binom{n+1}{2} + 6 \binom{n+1}{3} \\ &= 1 \cdot (n+1)/1! - 2 \cdot (n+1)n/2! + 6 \cdot (n+1)(n)(n-1)/3! \\ &= (n+1) [1 - n + n^2 - n] = (n+1)(n^2 - 2n + 1) = (n+1)(n-1)^2 \end{aligned}$$

4(a) The Stirling coefficients of the first kind are the unique integers $\{P_k\}$ such that $[n]_p = \sum_{k=0}^p (-1)^{p-k} \cdot \{P_k\} \cdot n^k$ for $0 \leq k \leq p$.

The Stirling coefficients of the second kind are the unique integers $\{P_k\}$ such that $n^k = \sum_{k=0}^p \{P_k\} \cdot [n]_k$, where $[n]_k = n(n-1)(n-2)\dots[n-(k-1)]$.

(b) From the definition above we know that

$$n^p = \sum_{k=0}^p \{P_k\} [n]_k = \{P_p\} [n]_p + \sum_{k=1}^{p-1} \{P_k\} \cdot [n]_k + \{P_0\} \cdot [n]_0 \dots (*)$$

$$\text{Now } n^p = n^{p-1} \cdot n = \sum_{k=0}^{p-1} \left\{ \begin{matrix} p-1 \\ k \end{matrix} \right\} \cdot [n]_k \cdot [(n-k) + k]$$

$$= \sum_{k=0}^{p-1} \left\{ \begin{matrix} p-1 \\ k \end{matrix} \right\} \cdot [n]_k \cdot (n-k) + \sum_{k=0}^{p-1} k \cdot \left\{ \begin{matrix} p-1 \\ k \end{matrix} \right\} \cdot [n]_k$$

$$= \left(\sum_{k=0}^{p-1} \left\{ \begin{matrix} p-1 \\ k \end{matrix} \right\} \cdot [n]_{k+1} \right) + \left(\sum_{k=1}^{p-1} k \cdot \left\{ \begin{matrix} p-1 \\ k \end{matrix} \right\} \cdot [n]_k \right) + 0 \cdot \left\{ \begin{matrix} p-1 \\ 0 \end{matrix} \right\} \cdot [n]_0$$

$$= \sum_{k=1}^p \left\{ \begin{matrix} p-1 \\ k-1 \end{matrix} \right\} \cdot [n]_k + \sum_{k=1}^{p-1} k \cdot \left\{ \begin{matrix} p-1 \\ k \end{matrix} \right\} \cdot [n]_k + 0 \cdot [n]_0$$

$$= \left\{ \begin{matrix} p-1 \\ p-1 \end{matrix} \right\} [n]_p + \sum_{k=1}^{p-1} \left(\left\{ \begin{matrix} p-1 \\ k-1 \end{matrix} \right\} + k \cdot \left\{ \begin{matrix} p-1 \\ k \end{matrix} \right\} \right) \cdot [n]_k + 0 \cdot [n]_0 \dots (**)$$

Equating the coefficients of $[n]_k$ for $1 \leq k \leq p-1$ we get in (*) & (**), we get $\{P_k\} = \left\{ \begin{matrix} p-1 \\ k-1 \end{matrix} \right\} + k \cdot \left\{ \begin{matrix} p-1 \\ k \end{matrix} \right\}$ for $1 \leq k \leq p-1$

[We also get $\{P_p\} = \left\{ \begin{matrix} p-1 \\ p-1 \end{matrix} \right\}$ and $\{P_0\} = 0$ by equating the coeff. of $[n]_p$ & $[n]_0$ - but this was not requested.]

5. Select one member of the group G & call her (or him) P_i . Let $A =$ set of acquaintances of P_i & $S =$ set of strangers to P_i in G . Then $|A \cup S| = 19$ and $A \cap S = \emptyset$. So either $|A| \geq 10$ or $|S| \geq 10$.

Case (i) $|A| \geq 10$. In this case we know by a theorem in class (which we are allowed to use) that A contains 3 mut. acq. or 4 mut. str. Now if A contains 3 mut. acq., add P_i to get 4 mut. acq. And if A contains 4 mut. strangers, we got what we wanted, namely 4 mut. strangers.

Case (ii) $|S| \geq 10$. In this case S contains 4 mut. acq. or 3 mut. strangers by the result we are also allowed to use. Now if S contains 4 mut. acq., we are done. And if S contains 3 mut. strangers, just add P_i to get 4 mut. strangers.

So in all cases we get either 4 mut. acq. or 4 mut. strangers.

6(a) $(E^2 + 2E + 2I)x_n = 0 \quad \therefore E = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm i$
 $\therefore x_n^c = A(-1+i)^n + B(-1-i)^n$. Try $x_n^p = a + bn$. Then
 $x_{n+1}^p = a + b(n+1)$ & $x_{n+2}^p = a + b(n+2)$. So $x_{n+2} + 2x_{n+1} + 2x_n = 10n$, becomes
 $(a + bn + 2b) + 2(a + bn + b) + 2(a + bn) = 10n$
 $\therefore (5a + 4b) + 5bn = 10n$. $\therefore b = 2$ & $5a + 4(2) = 0 \Rightarrow a = -\frac{8}{5}$
 $\therefore x_n^p = (-\frac{8}{5}) + 2n$. $\therefore x_n = x_n^c + x_n^p = A(-1+i)^n + B(-1-i)^n + 2n - \frac{8}{5}$.

(b) $(E^2 - E - 2I)x_n = 0$, so $(E+1)(E-2)x_n = 0$. So $x_n^c = A(-1)^n + B(2)^n$.
 Try $x_n^p = a \cdot n \cdot 2^n$ because 2 is a root of homog. eq. of multiplicity 1.
 Then $x_{n+1}^p = a \cdot (n+1) \cdot 2^{n+1} = 2a(n+1) \cdot 2^n$
 and $x_{n+2}^p = a(n+2) \cdot 2^{n+2} = 4a(n+2) \cdot 2^n$.

So $x_{n+2} - x_{n+1} - 2x_n = 6 \cdot 2^n$ becomes

$$(4an + 8a) \cdot 2^n - (2a \cdot n + 2a) \cdot 2^n - 2(a \cdot n) \cdot 2^n = 6 \cdot 2^n$$

$$\therefore (4a - 2a - 2a) \cdot n \cdot 2^n + (8a - 2a) \cdot 2^n = 6 \cdot 2^n$$

$$\therefore 0 + 6a \cdot 2^n = 6 \cdot 2^n \Rightarrow a = 1. \quad \therefore x_n^p = 1 \cdot n \cdot 2^n$$

$$\therefore x_n = x_n^c + x_n^p = A(-1)^n + B(2)^n + n \cdot 2^n. \quad \text{END}$$