

Answer all 6 questions. No Calculators or Cell phones are allowed. An unjustified answer will receive little credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

- (16) 1.(a) Find the *permutation* which has $\langle 4, 3, 3, 2, 1, 1, 0 \rangle$ as its *inversion sequence*.
(b) How many *integers* in $\{1, 2, 3, \dots, 700\}$ are *divisible* by neither 12 nor 15?
- (16) 2.(a) Write down what is the *Multinomial Theorem* in full details. Then use it to find the *coefficient of $x^3y^6z^2$* in the expansion of $(3x^3 - 2y^2 + z/6)^6$.
(b) Find the number of *integer solutions* of the equation: $x_1 + x_2 + x_3 + x_4 = 10$ with $x_1 \geq 2, x_2 \geq -3, x_3 \geq 4$, and $x_4 \geq 5$.
- (18) 3.(a) Write down *version 1* of the *Inclusion-Exclusion Theorem* in full details.
(b) Find the no. of *20-combinations* of the multi-set $F = [7.a, 9.b, 11.c, 13.d]$.
[You may leave your answer in terms of simplified Binomial coefficients.]
- (18) 4. (a) How many permutations of $\{3, 4, 5, 6, 7, 8, 9\}$ have *no even number going to an odd number and no odd number going to itself*.
(b) How many permutations of $\{3, 4, 5, 6, 7, 8, 9\}$ have *no prime number going to itself*? [You may leave your ans. in terms of simplified factorials.]
- (16) 5.(a) Let $n \in \mathbb{N}^+$ be *even*. Use the Binomial Theorem to give an *analytic proof* that $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n}$.
(b) Give a *combinatorial proof* of the same result stated in part (a).
- (16) 6.(a) Define what is an *ultimate set* with respect to the subsets A_1, A_2, \dots, A_n of U .
(b) Let $M = [n_1.a_1, n_2.a_2, \dots, n_k.a_k]$ be a *finite multi-set* & $n = n_1 + n_2 + \dots + n_k$. Prove that the number of *n-permutations* of M is $n!/(n_1!n_2! \dots n_k!)$.

1(a) $\langle 7 \rangle$ b/c $|S|=7$, (b) Let $U = \{1, 2, 3, \dots, 700\}$, then $|U| = 700$

$\langle 7, 6 \rangle$ b/c. $i_6=1$, $|A| = |\{x \in U : x \text{ is divisible by } 12\}| = \left\lfloor \frac{700}{12} \right\rfloor = 58$

$\langle 7, 5, 6 \rangle$ b/c. $i_5=1$, $|B| = |\{x \in U : x \text{ is divisible by } 15\}| = \left\lfloor \frac{700}{15} \right\rfloor = 46$

$\langle 7, 5, 4, 6 \rangle$ b/c. $i_4=2$, No. of x in U divisible by neither 12 nor 15

$\langle 7, 5, 4, 3, 6 \rangle$ b/c. $i_3=3$, $= |A^c \cap B^c| = |U| - |A| - |B| + |A \cap B|$

$\langle 7, 5, 4, 2, 3, 6 \rangle$ b/c. $i_2=3$, $= 700 - \left\lfloor \frac{700}{12} \right\rfloor - \left\lfloor \frac{700}{15} \right\rfloor + \left\lfloor \frac{700}{\text{lcm}(12, 15)} \right\rfloor$

Ans = $\langle 7, 5, 4, 2, 1, 3, 6 \rangle$ b/c. $i_1=4$, $= 700 - 58 - 46 + 11 = 711 - 104 = \boxed{607}$

2(a) Multinomial Theorem: For any $k, n \in \mathbb{N}$ we have $(x_1 + x_2 + \dots + x_k)^n$

$$= \sum_{\substack{(n_1, n_2, \dots, n_k) \\ (n_1, \dots, n_k) : n_1 + \dots + n_k = n}} (n_1, n_2, \dots, n_k) x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} \text{ where } (n_1, n_2, \dots, n_k) = \frac{n!}{n_1! n_2! \dots n_k!}.$$

So coefficient of $x^3 y^6 z^2$ in the expansion of $(3x^3 - 2y^2 + z/6)^6$

$$= \binom{6}{3, 2} \cdot (3)^1 \cdot (-2)^3 \cdot \left(\frac{1}{6}\right)^2 = \frac{6!}{1! 3! 2!} \cdot 3 \cdot (-8) \cdot \frac{1}{36} = \frac{8 \cdot 5 \cdot 4 \cdot 3}{*} \cdot \frac{3}{1} \cdot -8 \cdot \frac{1}{36} = \boxed{-40}$$

(b) Let $x_1 = y_1+2$, $x_2 = y_2-3$, $x_3 = y_3+4$, $x_4 = y_4+5$. Then no. of integer solutions of $x_1 + x_2 + x_3 + x_4 = 10$ with $x_1 \geq 2$, $x_2 \geq -3$, $x_3 \geq 4$ & $x_4 \geq 5$ = no. of integer solutions of $(y_1+2) + (y_2-3) + (y_3+4) + (y_4+5) = 10$ with $y_i \geq 0$ = no. of non-neg solutions of " $y_1 + y_2 + y_3 + y_4 = 2$ " = $\binom{2+4-1}{4-1} = \boxed{10}$.

3 (a) Let A_1, A_2, \dots, A_n be subsets of a universal set U . Then

$$|A_1^c \cap A_2^c \cap \dots \cap A_n^c| = \sum_{k=0}^n (-1)^k \left\{ \sum_{\langle i_1, \dots, i_k \rangle : 1 \leq i_1 < i_2 < \dots < i_k \leq n} |U \cap A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| \right\}.$$

(b) Let $M = \{\infty, a, \infty, b, \infty, c, \infty, d\}$, U = set of 20-combinations of M ,

$$A = \{x \in U : x \text{ has } \geq 7a's\}, \quad B = \{x \in U : x \text{ has } \geq 9b's\}, \quad |U| = \binom{20+4-1}{4-1},$$

$$C = \{x \in U : x \text{ has } \geq 11c's\} \text{ and } D = \{x \in U : x \text{ has } \geq 13d's\}. \quad \text{Then}$$

$$|A| = |\{x \in U : x \text{ has } \geq 8a's\}| = |\{x \cup [Ba] : x \text{ is a 12-comb. of } M\}| = \binom{12+4-1}{4-1}$$

$$\text{Similarly } B = |\{x \cup [0b] : x \text{ is a 10-comb. of } M\}| = \binom{10+4-1}{4-1} = \binom{13}{3}$$

$$|C| = |\{x \cup [2c] : x \text{ is an 8-comb. of } M\}| = \binom{8+4-1}{4-1} = \binom{11}{3}. \quad \text{Also}$$

$$3(b) |A| = |\{x \in [14d] : x \text{ is a 6-comb of } M\}| = \binom{16+4-1}{4-1} = \binom{9}{3}$$

$$|A \cap B| = |\{x \in [8a, 10b] : x \text{ is a 2-comb of } M\}| = \binom{2+4-1}{4-1} = \binom{5}{3}$$

$$|A \cap C| = |\{x \in [8a, 12c] : x \text{ is a 0-comb of } M\}| = \binom{0+4-1}{4-1} = \binom{3}{3}$$

$$A \cap D = B \cap C = B \cap D = C \cap D = \emptyset, \quad A \cap B \cap C = A \cap B \cap D = A \cap C \cap D = \emptyset,$$

$B \cap C \cap D = A \cap B \cap C \cap D = \emptyset$. So no. of 20-comb. of F

$$= |A^c \cap B^c \cap C^c \cap D^c| = |U| - |A| - |B| - |C| - |D| + |A \cap B| + |A \cap C| + |A \cap D| \\ + |B \cap C| + |B \cap D| + |C \cap D| - |A \cap B \cap C| - |A \cap B \cap D| - |A \cap C \cap D| - |B \cap C \cap D| + |A \cap B \cap C \cap D|$$

$$= \binom{23}{3} - \binom{15}{3} - \binom{13}{3} - \binom{11}{3} - \binom{9}{3} + \binom{5}{3} + \binom{3}{3}$$

4(a) No. of perm. of $\{3, 4, 5, 6, 7, 8, 9\}$ with no even \rightarrow an odd and
no odd \rightarrow itself

$$= \text{no. of perm with } \begin{pmatrix} 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \cancel{3} & \downarrow & \cancel{5} & \downarrow & \cancel{7} & \downarrow & \cancel{9} \\ 4, 6 \text{ or } 8 & & 5 & 4, 6 \text{ or } 8 & 7 & 4, 6 \text{ or } 8 & 9 \end{pmatrix}$$

$\{4, 6, 8\} \rightarrow \{4, 6, 8\}$ and $\{3, 5, 7, 9\}$ deranged

$$= (3!)(2!) = 6(9) = \boxed{54} \text{ b.c. } D_4 = 4D_3 + (-1)^4 = 4(2) + 1 = 9.$$

(b) Let U = set of perm. of $\{3, 4, 5, 6, 7, 8, 9\}$, (Here 3, 5 & 7 are prime)

$$A_3 = \{x \in U : 3 \rightarrow 3 \text{ in } x\}, \quad A_5 = \{x \in U : 5 \rightarrow 5 \text{ in } x\}, \text{ and}$$

$$A_7 = \{x \in U : 7 \rightarrow 7 \text{ in } x\}. \text{ Then } |U| = 7!, \quad |A_3| = |A_5| = |A_7| = 6!$$

$$(A_3 \cap A_5) = |A_3 \cap A_5| = |A_5 \cap A_7| = 5! \quad \text{and} \quad |A_3 \cap A_5 \cap A_7| = 4! \quad \text{So}$$

no. of permutations of $\{3, 4, 5, 6, 7, 8, 9\}$ with no prime \rightarrow itself

$$= |A_3^c \cap A_5^c \cap A_7^c| = |U| - |A_3| - |A_5| - |A_7| + |A_3 \cap A_5| + |A_3 \cap A_7| + |A_5 \cap A_7|$$

$$= 7! - 3(6!) + 3(5!) - 4! - |A_3 \cap A_5 \cap A_7|$$

5(a) Let $n \in \mathbb{N}^+$ be even. Then by the standard Binomial Theorem

$$(1+x)^n = \binom{n}{0} \cdot x^0 + \binom{n}{1} \cdot x^1 + \binom{n}{2} \cdot x^2 + \dots + \binom{n}{n-1} \cdot x^{n-1} + \binom{n}{n} \cdot x^n$$

Putting $x = -1$, we get:

$$0 = (1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots - \binom{n}{n-1} + \binom{n}{n}.$$

Taking all the negative terms to the L.H.S. we get

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n}.$$

[Note: The result does not hold for $n=0$ because $(1-1)^0 = 0^0 = 1$.]

5(b) Let $U = \{1, 2, 3, \dots, n\}$, $P_o(U) = \{A \subseteq U : |A| \text{ is odd}\}$,
 $P_e(U) = \{A \subseteq U : |A| \text{ is even}\}$. Then

$$|P_o(U)| = |\{A \subseteq U : |A|=1\}| + |\{A \subseteq U : |A|=3\}| + \dots + |\{A \subseteq U : |A|=n-1\}| \\ = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1}. \text{ Also}$$

$$|P_e(U)| = |\{A \subseteq U : |A|=0\}| + |\{A \subseteq U : |A|=2\}| + \dots + |\{A \subseteq U : |A|=n\}| \\ = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n}. \text{ Now define } f: P_o(U) \rightarrow P_e(U)$$

by putting $f(A) = \begin{cases} A \cup \{n\} & \text{if } n \notin A \\ A - \{n\} & \text{if } n \in A \end{cases}$. Then f is a

well-defined function and $|f(A)|$ is even whenever $|A|$ is odd.

Also $f(f(A)) = \begin{cases} (A \cup \{n\}) - \{n\} = A & \text{if } n \notin A \\ ((A - \{n\}) \cup \{n\}) = A & \text{if } n \in A \end{cases}$ so $f \circ f = \text{identity}$
and thus $f^{-1} = f$.

Hence f is a bijection and hence $|P_o(U)| = |P_e(U)|$ b.c. $f[P_o(U)] = P_e(U)$.
Thus $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{n-1} = |P_o(U)| = |P_e(U)| = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{n}$.

6(a) An ultimate set w.r.t. the subsets A_1, A_2, \dots, A_n of U
is any expression of the form $X_1 n X_2 n \dots n X_n$ where $X_i = A_i$ or A_i^c .

(b) An n -perm. of M is an n -tuple of all the elements of M

$$\langle \underset{\uparrow}{a_2}, \underset{\uparrow}{a_1}, \underset{\uparrow}{a_1}, \underset{\text{III}}{a_3}, \dots, \underset{\uparrow}{a_2}, \underset{\uparrow}{a_1}, \underset{\text{III}}{a_3} \rangle$$

Now there $\binom{n}{n_1}$ ways to first place the n_1 a_1 's in n spaces,
 $\binom{n-n_1}{n_2}$ ways to place the n_2 a_2 's in $(n-n_1)$ available spaces,
 $\binom{n-n_1-n_2}{n_3}$ ways to place the n_3 a_3 's in $(n-n_1-n_2)$ available spaces,

$\binom{n-n_1-n_2-\dots-n_{k-1}}{n_k}$ ways to place n_k a_k 's in the last n_k spaces.

So number of n -permutations of M

$$= \binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} \cdot \dots \cdot \binom{n-n_1-n_2-\dots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdot \frac{(n-n_1-n_2)!}{n_3!} \cdot \dots \cdot \frac{(n-n_1-\dots-n_{k-1})!}{n_k!(n-n_1-\dots-n_{k-1})!} \cdot \frac{(n-n_1-\dots-n_k)!}{n_k! 0!}$$

$$= \frac{n!}{n_1! n_2! n_3! \dots n_{k-1}! n_k!} = (n!) / (n_1! n_2! \dots n_k!).$$