(15) 1. Find the solution of the recurrence equation \( x_{n+2} - 2x_{n+1} - 8x_n = 0 \) with the initial conditions \( x_0 = 5 \) and \( x_1 = 2 \).

(20) 2. Find the complementary solution and give (with reasons) the minimum form of a particular solution of each of the following recurrence equations.
(a) \((E^2 + E - 2I)(E^2 + I) (x_n) = 4n^2\).
(b) \((E^2 - 2E + 2I)(E^2 - 4E + 4I) (x_n) = 5n(2)^n\).

(20) 3(a) Use the method of generating functions to find the solution of the recurrence equation: \( a_n + 3a_{n-1} - 4 = 0 \) with \( a_0 = 3 \).
(b) Let \( M = \{ \infty.a, \infty.b, 9.c \} \). Find the generating function of the set of all \( n \)-combinations of \( M \) with at least four \( a \)'s, an even numbers of \( b \)'s, and an odd numbers of \( c \)'s. [Simplify your answer as far as possible.]

(15) 4(a) Define what are the operators \( E \) and \( \Delta \) that operates on a sequence \( \langle x_n \rangle \).
(b) Let \( h_k = 3k^2 - 3k + 1 \). Find the zero-column of \( \langle h_k \rangle_{k \in \mathbb{N}} \) & use it to get a formula for the sum \( h_0 + h_1 + h_2 + \ldots + h_n \). [Simplify your answer as far as possible.]

(15) 5. (a) Write down what is the function form of the Pigeon-Hole Principle.
(b) Let \( \langle a_1, a_2, a_3, \ldots, a_n \rangle \) be any sequence of \( n \) integers. Prove that there is a non-empty segment of consecutive terms of this sequence whose sum is a multiple of \( n \).

(15) 6(a) Define what are the Stirling coefficients of the First & Second kinds.
(b) Prove for any \( k \) with \( 0 < k < p \), the Stirling coefficientss of the first kind satisfy
\[
\left[ \binom{p}{k} \right] = \left[ \binom{p - 1}{k - 1} \right] + (p - 1) \left[ \binom{p - 1}{k} \right].
\]
1. \[x_{n+2} - 2x_{n+1} - 8x_n = 0 \text{ with } x_0 = 5 \text{ and } x_1 = 2. \text{ This becomes }\]
\[(E^2 - 2E - 8I)X_n = 0 \implies (E+2I)(E-4I)X_n = 0. \text{ So } E = -2 \text{ or } 4.\]
\[\therefore X_n = A \cdot (-2)^n + B \cdot (4)^n. \text{ Now } x_0 = 5 \implies 5 = A + B \]
\[\therefore B = 5 - A. \text{ And } x_1 = 2 \implies 2 = -2A + 4B \]
\[\therefore 2 = -2A + 4(5-A) \implies 6A = 18 \implies A = 3. \therefore B = 5 - 3 = 2.\]
\[\text{So } X_n = 3 \cdot (-2)^n + 2 \cdot (4)^n. \text{ Check: } x_0 = 3(-2)^0 + 2(4)^0 = 5 \]
\[x_1 = 3(-2)^1 + 2(4)^1 = 2\]

2. (a) \[(E^2 + E - 2I)(E^2 + I)X_n = 0 \implies (E-I)(E+2I)(E-2I)(E+I)X_n = 0.\]
\[\therefore E = 1, -2, i, \text{ or } -i. \text{ So complementary solution is given by } X^c_n = A \cdot (i)^n + B \cdot (-2)^n + C \cdot (i)^n + D \cdot (-i)^n.\]
\[(E-I)(E+2I)(E-2I)(E+I)X_n = 4n^2 = 4n^2(1)^n. \text{ Since } 4n^2 \text{ is a polynomial of degree } 2 \text{ and } 1 \text{ is a root of the auxiliary equation of multiplicity } 1,\]
the minimal form of a particular solution will be
\[X^p_n = (a + bn + cn^2) \cdot n^2 \cdot (1)^n = an + bn^2 + cn^3.\]

(b) \[(E^2 - 2E + 2I)(E^2 - 4E + 4I)X_n = 0 \implies (E^2 - 2E + 2I)(E-2I)^2X_n = 0.\]
\[E^2 - 2E + 2I = 0 \implies E = \frac{(-2) + \sqrt{(-2)^2 - 4(2)(2)}}{2} = 1 \pm i.\]
So the roots of the auxiliary equations are 2, 2, 1+i, 1-i.
\[\therefore X^c_n = A \cdot (2)^n + Bn \cdot (2)^n + C \cdot (1+i)^n + D \cdot (1-i)^n.\]
\[(E^2 - 2E + 2I)(E-2I)^2X_n = 5n \cdot (2)^n. \text{ Since } 5n \text{ is a polynomial of degree } 1 \text{ and } 2 \text{ is a root of the auxiliary equation of multiplicity } 2, \text{ the minimal form of a particular solution will be}\]
\[X^p_n = (a + bn) \cdot n \cdot (2)^n = an \cdot (2)^n + bn^2 \cdot (2)^n.\]
Notice in both (a) & (b) \(X^c_n \& X^p_n\) have no terms in common. [A term is an expr. like \(bn \cdot (2)^n\) or \(an^2 \cdot (2)^n\).]
3(a) let \( f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \). Then \( 3x \cdot f(x) = 3a_0x + 3a_1x^2 + \ldots + 3a_{n-1}x^n + \ldots \). Also \( \frac{-4}{1-x} = -4 - 4x - 4x^2 - \ldots - 4x^n - \ldots \). 

\[ \therefore (1+3x) \frac{-4}{1-x} = (a_0-4) + (a_1+3a_0-4)x + \ldots + (a_{n-3}+3a_{n-4}-4)x^n + \ldots \]

\[ = (3-4) + 0.x + 0.x^2 + \ldots + 0.x^n + \ldots = -1 \]

\[ \therefore (1+3x) f(x) = 1 + 4/(1-x) = (3+x)/(1-x) \]

\[ \therefore f(x) = \frac{3+x}{1-x} = \frac{A}{1-x} + \frac{B}{1+x} \]

\[ A + B = 3, \quad \text{and} \quad 3A - B = 1 \Rightarrow A = 3A - 1 \Rightarrow 4A = 4 \]

\[ \therefore f(x) = \frac{1}{1-x} + \frac{3}{1-(3x)} = \frac{1}{1-x} \left[ 1 + x + x^2 + \ldots + x^n + \ldots \right] \]

\[ + 2 \left[ 1 + (3x) + (3x)^2 + \ldots + (3x)^n + \ldots \right] \]

\[ \therefore a_n = \text{coeff. of } x^n \text{ in the exp. of } f(x) = 1 + 2(-3)^n. \text{ Check } a_0 = 1 + 2 = 3. \]

(b) see end of solutions — after 6(b).

4(a) The operator \( E \) is defined by \( E(\langle x_n \rangle_{n \in \mathbb{N}}) = \langle x_{n+1} \rangle_{n \in \mathbb{N}} \).

The operator \( \Delta \) is defined by \( \Delta(\langle x_n \rangle_{n \in \mathbb{N}}) = \langle x_{n+1} - x_n \rangle_{n \in \mathbb{N}} \).

(b) \( \langle \Delta^k \rangle_{n \in \mathbb{N}} \)

\[
\begin{array}{ccccccc}
\langle h_k \rangle & 1 & 1 & 7 & 19 & 37 & 61 \\
\langle \Delta h_k \rangle & 0 & 6 & 12 & 18 & 24 & \\
\langle \Delta^2 h_k \rangle & 6 & 6 & 6 & 6 & \\
\langle \Delta^3 h_k \rangle & 0 & 0 & 0 & 0 & \\
\end{array}
\]

So zero-column of \( \langle h_k \rangle \) is \( \langle 1, 0, 6, 0, 0, \ldots \rangle \).

Thus \( h_k = 1 \cdot k + 0 \cdot \binom{k}{1} + 6 \cdot \binom{k}{2} \). Hence

\[
\sum_{k=0}^{n} h_k = 1 \cdot \sum_{k=0}^{n} k + 0 \cdot \sum_{k=0}^{n} \binom{k}{1} + 6 \cdot \sum_{k=0}^{n} \binom{k}{2} = 1 \cdot \frac{n(n+1)}{2} + 0 \cdot \frac{n(n+1)}{2} + 6 \cdot \frac{n(n+1)(2n+1)}{6} \\
= \frac{n(n+1)}{2} + 0 + 6 \frac{n(n+1)(2n+1)}{6} = \frac{n^3 + 1}{3} \]
5(b) \[ f(x) = f(x_1). \] Note when \( i = 0, \quad a_i + \ldots + a_i = \sum_{i \in \emptyset} a_i = 0. \]

Thus \( a_i + \ldots + a_i + a_i + \ldots + a_j = a_i + \ldots + a_j \pmod{n}. \)

Hence \( a_i + a_i + \ldots + a_j = 0 \pmod{n}. \) So the sum of the terms in the segment \( \langle a_{i+1}, a_{i+2}, \ldots, a_{j-1}, a_j \rangle \) will be an integer multiple of \( n. \) Since \( i < j, \) this segment will be non-empty. Note when \( i = 0, \quad \langle a_{i+1}, \ldots, a_j \rangle = \langle a_1, a_2, \ldots, a_j \rangle \) and when \( i = j - 1, \quad \langle a_{i+1}, a_{i+2}, \ldots, a_j \rangle = \langle a_{i+1} \rangle = \langle a_j \rangle. \)

6(a) The Stirling coefficients of the first kind are defined to be the unique integers \( \left\{ P \right\} \) such that \( [n]_p = \sum_{k=0}^{n} (-1)^{n-k} \left\{ P \right\} [k]. \)

The Stirling coefficients of the second kind are defined to be the unique integers \( \left\{ \frac{k}{n} \right\} \) such that \( n_p = \sum_{k=0}^{n} \left\{ \frac{k}{n} \right\} [k]. \)

Here \( \left\{ \frac{n}{n-1} \right\} = (n-1)(n-2) \ldots (n-(p-1)). \)

(b) We know that \( [n]_p = (-1)^{p-1} [p] p^n + \sum_{k=1}^{p-1} (-1)^{p-k} \left\{ \frac{k}{n} \right\} [k] n^k + (-1)^{p-1} \left\{ \frac{p}{n} \right\} n^{p-1}. \)

So \( [n]_p = [n]_{p-1} [n-(p-1)] = \sum_{k=0}^{p-1} (-1)^{p-k} \left\{ \frac{k}{n-1} \right\} [k] n^{p-1}. \)

\[ \sum_{k=0}^{p-1} (-1)^{p-k} \left\{ \frac{k}{n-1} \right\} n^{p-1} = (p-1) \sum_{k=0}^{p-1} (-1)^{p-k-1} \]

\[ \sum_{k=0}^{p-1} (-1)^{p-k} \left\{ \frac{k}{n-1} \right\} n^{k+1} = (p-1) \sum_{k=0}^{p-1} (-1)^{p-k-1} \]

\[ \sum_{k=1}^{p-1} (-1)^{p-k} \left\{ \frac{k}{n-1} \right\} n^{k+1} = (p-1) \sum_{k=0}^{p-1} (-1)^{p-k-1} \]

\[ \sum_{k=0}^{p-1} (-1)^{p-k} \left\{ \frac{k}{n-1} \right\} n^{p-1} = (p-1) \sum_{k=0}^{p-1} (-1)^{p-k-1} \]

Evaluating the coefficients of \((x)\) with the last equation, we get \( \left\{ \frac{p}{k} \right\} = \left\{ \frac{p-1}{k-1} \right\} + (p-1). \left\{ \frac{p-1}{k} \right\} \) for \( 1 \leq k \leq p-1. \)

Oops! I forgot to do Problem #3(b).

3(b) Gen. func. = \( (x^4 + x^5 + x^6 + \ldots) (x^0 + x^2 + x^4 + \ldots) (x^1 + x^3 + \ldots + x^9) \)

\[ = x^4 \left( 1 + x + x^2 + \ldots \right) \left( 1 + x^2 + x^4 + \ldots \right) \cdot x \left( 1 + x^2 + x^4 + x^6 + x^8 \right) \]

\[ = x^5 \cdot \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1-x^10}{1-x^2} = \frac{x^5 (1-x^{10})}{(1-x)^2 (1+x)^2} \]