

Answer all 6 questions. No Calculators or Cell phones are allowed. An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

- (16) 1.(a) How many *integers* in $\{1, 2, 3, \dots, 800\}$ are *divisible* by *neither* 10 nor 12?
(b) Find the *permutation* which has $\langle 3, 2, 3, 0, 1, 0 \rangle$ as its *inversion sequence* and check that your answer is correct.
- (16) 2.(a) Write down what is the *Multinomial Theorem* in full details. Then use it to find the *coefficient of* $x^4y^3z^3$ in the expansion of $(3x^2 + 4y^3 - z/3)^6$.
(b) Find the number of *integer solutions* of the equation: $x_1 + x_2 + x_3 + x_4 = 9$ with $x_1 \geq 2, x_2 \geq -4, x_3 \geq 5, \text{ and } x_4 \geq 3$.
- (16) 3.(a) Write down *version 1* of the *Inclusion-Exclusion Theorem* in full details.
(b) Find the number of *20-combinations* of the multi-set $F = [7.a, 9.b, 18.c]$.
[You should leave your answer in terms of simplified Binomial coefficients.]
- (18) 4. (a) How many permutations of $\{2, 3, 4, 5, 6, 7, 8\}$ have *each odd number going to an odd number and no even number going to itself*?
(b) How many permutations of $\{2, 3, 4, 5, 6, 7, 8\}$ have *no odd number going to itself*? [You may leave your answer in terms of simplified factorials.]
- (18) 5.(a) Give a *combinatorial* proof of Pascal's identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, for all integers k and n with $1 \leq k \leq n-1$.
(b) Prove that for $0 \leq k \leq n$, $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}$.
- (16) 6.(a) Define what is a *positive set* and what is an *ultimate set* with respect to the subsets A_1, A_2, \dots, A_n of U .
(b) Let $M = [\infty.a_1, \infty.a_2, \dots, \infty.a_k]$ be a *multi-set with infinite numbers of k kinds of elements*. Prove that the number of *r-combinations* of M is $\binom{r+k-1}{k-1}$.

1(a) Let $U = \{1, 2, 3, \dots, 800\}$, $A = \{x \in U : x \text{ is divisible by } 10\}$, and

$B = \{x \in U : x \text{ is divisible by } 12\}$. Then our answer will be $|A^c \cap B^c|$

$$\begin{aligned} |U| - |A| - |B| + |A \cap B| &= 800 - \left\lfloor \frac{800}{10} \right\rfloor - \left\lfloor \frac{800}{12} \right\rfloor + \left\lfloor \frac{800}{\text{lcm}(10, 12)} \right\rfloor \\ &= 800 - 80 - 66 + \left\lfloor \frac{800}{60} \right\rfloor = 654 + 13 = \boxed{667} \end{aligned}$$

(b) Write down $\langle 6 \rangle$ because the given seq. has length 6.

Then $\langle 6, 5 \rangle$ because $i_5 = 1$

$\langle 4, 6, 5 \rangle$ because $i_4 = 0$

$\langle 4, 6, 5, 3 \rangle$ because $i_3 = 3$

$\langle 4, 6, 2, 5, 3 \rangle$ because $i_2 = 2$

$\langle 4, 6, 2, 1, 5, 3 \rangle$ because $i_1 = 3$. inversion sequence

Check: $\{ = \langle 3, 2, 3, 0, 1, 0 \rangle \checkmark$

2(a) Let $k, n \in \mathbb{N}$. Then $(x_1 + x_2 + \dots + x_k)^n = \sum (n_1, n_2, \dots, n_k) x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$

where $(n_1, n_2, \dots, n_k) = \frac{n!}{n_1! n_2! \dots n_k!}$. $\langle n_1, \dots, n_k \rangle : n_1 + n_2 + \dots + n_k = n$.

The term $x^4 y^3 z^3$ comes from the term $\binom{6}{2, 1, 3} \cdot (3x^2)^2 \cdot (4y^2) \cdot (-z/3)^3$

in the expansion of $(3x^2 + 4y^2 - z/3)^6$. So our answer will be

$$\binom{6}{2, 1, 3} \cdot (3)^2 \cdot (4)^1 \cdot \left(-\frac{1}{3}\right)^3 = \frac{6!}{2! 1! 3!} \cdot 3^2 \cdot 4 \cdot \frac{-1}{3^3} = \frac{-6 \cdot 5 \cdot 4 \cdot 3!}{2 \cdot 3!} \cdot \frac{3^2 \cdot 4}{3^3} = \boxed{-80}$$

(b) Let $x_1 = y_1 + 2$, $x_2 = y_2 - 4$, $x_3 = y_3 + 5$, and $x_4 = y_4 + 3$. Then

No. of integer solutions of $x_1 + x_2 + x_3 + x_4 = 9$ with $x_1 \geq 2$, $x_2 \geq -4$, $x_3 \geq 5$, $x_4 \geq 3$

= no. of integer solutions of $(y_1 + 2) + (y_2 - 4) + (y_3 + 5) + (y_4 + 3) = 0$

with $y_i \geq 0$ = no. of non-neg. solutions of $y_1 + y_2 + y_3 + y_4 = 3$

$$= \binom{3+4-1}{4-1} = \binom{6}{3} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! 3!} = 5(4) = \boxed{20}$$

3(a) Let A_1, A_2, \dots, A_n be subsets of a universal set U . Then

$$|A_1^c \cap A_2^c \cap \dots \cap A_n^c| = \sum_{k=0}^n (-1)^k \left\{ \sum_{\langle i_1, \dots, i_k \rangle : 1 \leq i_1 < i_2 < \dots < i_k \leq n} |\cup_{i=1}^k A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| \right\}$$

3(b) Let $M = \{\infty, a, \infty, b, \infty, c\}$ and \mathcal{U} = set of all 20-combinations of M ,

$A = \{x \in \mathcal{U} : x \text{ has } > 7a's\}, B = \{x \in \mathcal{U} : x \text{ has } > 9b's\}, C = \{x \in \mathcal{U} : x \text{ has } > 18c's\}$

$$\text{Then } |A| = |\{x \in \mathcal{U} : x \text{ has } \geq 8a's\}| = |\{x \in [8, a] : x \text{ is a 12-comb. of } M\}| = \binom{12+3-1}{3-1}$$

$$\text{Similarly } |B| = |\{x \in [10, b] : x \text{ is a 10-comb. of } M\}| = \binom{10+3-1}{3-1} = \binom{12}{2}$$

$$\text{and } |C| = |\{x \in [C] : x \text{ is a 1-comb. of } M\}| = \binom{1+3-1}{3-1} = \binom{3}{2}. \text{ Also}$$

$$|A \cap B| = |\{x \in [8a, 10b] : x \text{ is a 2-comb. of } M\}| = \binom{2+3-1}{3-1} = \binom{4}{2} \text{ and}$$

$$A \cap C = \emptyset, B \cap C = \emptyset \text{ and } A \cap B \cap C = \emptyset, \text{ and } |\mathcal{U}| = \binom{20+3-1}{3-1} = \binom{22}{2}$$

$$\therefore \text{Answer} = |A^c \cap B^c \cap C^c| = |\mathcal{U}| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| \\ = \binom{22}{2} - \binom{14}{2} - \binom{12}{2} - \binom{3}{2} + \binom{4}{2}$$

4(a) No. of perm. of $\{2, 3, 4, 5, 6, 7, 8\}$ with each odd number going to an odd number and with no even number going to itself

= no. of perm. with $\{3, 5, 7\} \rightarrow \{3, 5, 7\}$ and $\{2, 4, 6, 8\}$ deranged
(because 2, 4, 6, 8 cannot go to any odd no. since odds \rightarrow odds)

= (no. of perm. of $\{3, 5, 7\}$) · (no. of derangements of $\{2, 4, 6, 8\}$)

$$= (3!) D_4 = 6(9) = 54 \text{ because } D_4 = 4(D_3) + (-1)^4 = 9.$$

(b) Let \mathcal{U} = set of perm. of $\{2, 3, 4, 5, 6, 7, 8\}$ and put

$$A_3 = \{x \in \mathcal{U} : 3 \rightarrow 3 \text{ in } x\}, A_5 = \{x \in \mathcal{U} : 5 \rightarrow 5 \text{ in } x\},$$

$$A_7 = \{x \in \mathcal{U} : 7 \rightarrow 7 \text{ in } x\}. \text{ Then } |\mathcal{U}| = 7!, |A_3| = |A_5| = |A_7| = 6!$$

$$|A_3 \cap A_5| = |A_3 \cap A_7| = |A_5 \cap A_7| = 5! \text{ and } |A_3 \cap A_5 \cap A_7| = 4!$$

So no. of perm. of $\{2, 3, 4, 5, 6, 7, 8\}$ with no odd no. \rightarrow itself

$$= |A_3^c \cap A_5^c \cap A_7^c| = |\mathcal{U}| - |A_3| - |A_5| - |A_7| + |A_3 \cap A_5| + |A_5 \cap A_7| + |A_3 \cap A_7| \\ = 7! - 3(6!) + 3(5!) - 4! - |A_3 \cap A_5 \cap A_7|$$

5(a) $\binom{n}{k}$ = No. of k -subsets of $\{1, 2, 3, \dots, n\}$ (with $n \geq 1$)

= No. of k -subsets of $\{1, 2, 3, \dots, n\}$ not containing "1"

+ No. of k -subsets of $\{1, 2, 3, \dots, n\}$ containing "1"

= No. of k -subsets of $\{2, 3, 4, \dots, n\}$ = $\binom{n-1}{k} + \binom{n-1}{k-1}$

+ No. of $(k-1)$ -subsets of $\{2, 3, 4, \dots, n\}$ = $\binom{n-1}{k-1} + \binom{n-1}{k-1}$

(just add "1" to the $(k-1)$ -subset to get a k -subset containing "1".)

5(b) We will prove the result by parametric induction on n (k will be the temporarily fixed parameter).

Basis: For $n=k$, we have $\binom{k}{k} = 1 = \binom{k+1}{k+1} = \binom{n+1}{k+1}$. So the result is true for k .

Ind. Step: Suppose the result is true for n (where $n \geq k$). Then

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}.$$

$$\text{So } \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} + \binom{n+1}{k} = \binom{n+1}{k+1} + \binom{n+1}{k} = \binom{(n+1)+1}{k+1}$$

by Pascal's identity. So if the result is true for n , then it will be true for $n+1$. So by the Principle of Math Induction, the result is true for all $n \geq k$. Since k was arbitrary, the result is also true for all k .

6(a) A positive set w.r.t. the subsets A_1, \dots, A_n of U is any expression of the form $U \cap A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}$ where (i_1, i_2, \dots, i_k) is a subsequence of $(1, 2, 3, \dots, n)$ with $0 \leq k \leq n$.

An ultimate set w.r.t. A_1, \dots, A_n and U is any expression of the form $X_1 \cap X_2 \cap \dots \cap X_n$ where $X_i = A_i$ or A_i^c .

(b) First observe that the number of non-neg. integer solutions of $x_1 + x_2 + \dots + x_k = r$ is the number of ways of arranging r 1's and $(k-r)$ +'s in a row because each solution of $x_1 + \dots + x_k = r$ corresponds to one such arrangement. Now

No. of r -comb. of M = no. of sub-multisets of M of the form

$[x_1, q_1, x_2, q_2, \dots, x_k, q_k]$ with $x_1 + x_2 + \dots + x_k = r$ & $x_i \geq 0$

= no. of non-neg. integer solutions of $x_1 + \dots + x_k = r$

= no. of ways of arranging r 1's and $(k-r)$ +'s in a row

= no. of permutations of $[r, "1", (k-r), "+"]$

$$= \frac{[r+(k-r)]!}{r!(k-r)!} \quad (\text{by a theorem from class}) = \binom{r+k-1}{k-1}$$