

Answer all 6 questions. No notes, calculators or cellphones are allowed. An unjustified answer will receive little or no credit. BEGIN EACH OF THE QUESTIONS ON 6 SEPARATE PAGES.

- (15) 1. Find the solution of the recurrence equation $x_{n+2} + 2x_{n+1} - 8x_n = 0$ with the initial conditions $x_0 = 4$ and $x_1 = 2$.
- (20) 2. Find the complementary solution and give (with reasons) the minimum form of a particular solution of each of the following recurrence equations.
(a) $(E^2 + 2E - 3I)(E^2 + 4I)(x_n) = 5n^2$.
(b) $(E^2 - 2E + 5I)(E^2 - 6E + 9I)(x_n) = 2n \cdot (3)^n$.
- (20) 3(a) Use the method of generating functions to find the solution of the recurrence equation: $a_n + 2a_{n-1} - 9 = 0$ with $a_0 = 7$.
(b) Let $M = [\infty.a, 8.b, \infty.c]$. Find the generating function of the set of all n -combinations of M with at least four a's, an odd numbers of b's, and an even numbers of c's. [Simplify your answer as far as possible.]
- (15) 4(a) Define what are the operators E and Δ that operates on a sequence $\langle x_n \rangle_{k \in \mathbb{N}}$.
(b) Let $h_k = 3k^2 - 2k + 4$. Find the zero-column of $\langle h_k \rangle_{k \in \mathbb{N}}$ and use it to get a formula for the sum $h_0 + h_1 + h_2 + \dots + h_n$.
[Simplify your answer as far as possible.]
- (15) 5. (a) Write down what is the function form of the Pigeon-Hole Principle.
(b) Let P be a set of 10 people. Prove that we can always find 3 mutual acquaintances in P or 4 mutual strangers in P .
(You may use the fact that in any set of 6 people, we can find 3 mutual acquaintances or 3 mutual strangers.)
- (15) 6(a) Define what are the Stirling coefficients of the First & Second kinds.
(b) Prove for any k with $0 < k < p$, the Stirling coefficients of the Second kind satisfy $\begin{Bmatrix} p \\ k \end{Bmatrix} = \begin{Bmatrix} p-1 \\ k-1 \end{Bmatrix} + k \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix}$.

#1. We have $x_{n+2} + 2x_{n+1} - 8x_n = 0$ with $x_0 = 4$ & $x_1 = 2$. This becomes

$$(E^2 + 2E - 8I)x_n = 0 \Rightarrow (E - 2I)(E + 4I) = 0. \text{ So } E = -4 \text{ or } 2.$$

$$\therefore x_n = A.(2)^n + B.(-4)^n. \text{ Now } x_0 = 4 \Rightarrow 4 = A + B \Rightarrow B = 4 - A$$

$$x_1 = 2 \Rightarrow 2 = 2A - 4B$$

$$\therefore 2 = 2A - 4(4 - A) \Rightarrow 18 = 6A \Rightarrow A = 3. \therefore B = 4 - A = 4 - 3 = 1$$

$$\therefore x_n = 3(2)^n + (-4)^n. \text{ Check: } x_0 = 3(2)^0 + (-4)^0 = 4, x_1 = 3(2)^1 + (-4)^1 = 2 \checkmark$$

#2 (a) $(E^2 + 2E - 3I)(E^2 + 4I)x_n = 0 \Rightarrow (E - I)(E + 3I)(E - 2iI)(E + 2iI)x_n = 0$

$$\therefore E = 1, -3, 2i, \text{ or } -2i. \text{ So } x_n^c = A.(1)^n + B.(-3)^n + C.(2i)^n + D.(-2i)^n.$$

Now $(E - I)(E + 3I)(E - 2iI)(E + 2iI)x_n = 5.n^2 = 5.(n^2 + on + o).n^n$. Since $5n^2$ is a polynomial of degree 2, and 1 is a root of the auxiliary equation of multiplicity 1, the minimal form of a particular solution will be $x_n^p = (a + bn + cn^2).n^1.(1)^n = an + bn^2 + cn^3$.

(b) $(E^2 - 2E + 5I)(E^2 - 6E + 9I)x_n = 0 \Rightarrow (E^2 - 2E + 5I)(E - 3I)^2 x_n = 0$.

$$E^2 - 2E + 5I = 0 \Rightarrow E = [(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}]/2 = 1 \pm 2i.$$

So $E = 1+2i, 1-2i, 3$, and 3. Hence the complementary solution will be $x_n^c = A.(1+2i)^n + B.(1-2i)^n + (C + Dn).(3)^n$.

$(E^2 - 2E + 5I)(E - 3I)^2 x_n = 2n.(3)^n = (2n+o).(3)^n$. Since $2n$ is a polynomial of degree 1, and 3 is a root of the auxiliary equation of multiplicity 2, the minimal form of a particular solution will be $x_n^p = (a + bn).n^2.(3)^n = (an^2 + bn^3).3^n$.

[Notice in both (a) & (b), x_n^c & x_n^p have no terms in common.
So this is a way to partially check your answer.]

#3(a). Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

Then $2x f(x) = 2a_0 x + 2a_1 x^2 + \dots + 2a_{n-1} x^n + \dots$

and $(-9)/(1-x) = -9 - 9x - 9x^2 - \dots - 9x^n - \dots$

$$\begin{aligned} \therefore (1+2x)f(x) - 9/(1-x) &= (a_0 - 9) + (a_1 + 2a_0 - 9) + \dots + (a_n + 2a_{n-1} - 9) + \dots \\ &= (7-9) + 0 + \dots + 0 + \dots = -2. \end{aligned}$$

$$\therefore (1+2x)f(x) = 9/(1-x) + 2 = [9 - 2(1-x)]/(1-x) = \frac{7+2x}{1-x}$$

$$\therefore f(x) = \frac{7+2x}{(1-x)(1+2x)} = \frac{A}{1-x} + \frac{B}{1+2x}$$

$$\therefore 7+2x = A(1+2x) + B(1-x)$$

$$\text{Putting } x=1, \text{ gives us } 7+2(1) = A(1+2) + 0 \Rightarrow A=3.$$

$$\text{Putting } x=-\frac{1}{2}, \text{ gives us } 7+2(-\frac{1}{2}) = 0 + B(1-\frac{1}{2}) \Rightarrow B=4$$

$$\therefore f(x) = \frac{3}{1-x} + \frac{4}{1+2x} = 3[1+x+x^2+\dots+x^n+\dots] + 4[1+(-2x)+(2x)^2+\dots+(-2x)^n+\dots]$$

$$\therefore a_n = \text{coeff. of } x^n \text{ in the exp. of } f(x) = 3 + 4 \cdot (-2)^n$$

(b) Generating function of the number of n -combinations of M

$$\begin{aligned} &= (x^4 + x^5 + x^6 + \dots)(x + x^3 + x^5 + x^7)(1 + x^2 + x^4 + \dots) \\ &= x^4 (1 + x + x^2 + \dots) \cdot x \cdot (1 + x^2 + x^4 + x^6) \cdot (1 + x^2 + x^4 + \dots) \\ &= x^5 \cdot \frac{1}{1-x} \cdot \frac{1-(x^2)^4}{1-x^2} \cdot \frac{1}{1-x^2} = \frac{x^5 \cdot (1-x^8)}{(1-x)^3 (1+x)^2}. \end{aligned}$$

$$\# 4(a) E(\langle x_n \rangle_{n \in \mathbb{N}}) = \langle x_{n+1} \rangle_{n \in \mathbb{N}}, \quad \Delta(\langle x_n \rangle_{n \in \mathbb{N}}) = \langle x_{n+2} - x_n \rangle_{n \in \mathbb{N}}$$

$$(b) \langle k \rangle_{k \in \mathbb{N}} \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$$

$$\langle \Delta^0 h_k \rangle = \langle h_k \rangle_{k \in \mathbb{N}} \quad \begin{array}{|c|} \hline 4 \\ \hline \end{array} \quad 5 \quad 12 \quad 25 \quad 44 \quad \dots \quad h_k = 3k^2 - 2k + 4$$

$$\langle \Delta^1 h_k \rangle \quad \begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad 7 \quad 13 \quad 19 \quad \dots \quad \dots$$

$$\langle \Delta^2 h_k \rangle \quad \begin{array}{|c|} \hline 6 \\ \hline \end{array} \quad 6 \quad 6 \quad \dots \quad \dots \quad \dots$$

$$\langle \Delta^3 h_k \rangle \quad \begin{array}{|c|} \hline 0 \\ \hline \end{array} \quad 0 \quad \dots \quad \dots \quad \dots \quad \dots$$

So zero-column of $\langle h_k \rangle$ is $\langle 4, 1, 6, 0, 0, \dots \rangle$

$$\begin{aligned} \therefore h_k &= 4 \cdot \binom{k}{0} + 1 \cdot \binom{k}{1} + 6 \cdot \binom{k}{2}. \text{ Hence } \sum_{k=0}^n h_k = 4 \cdot \sum_{k=0}^n \binom{k}{0} + 1 \cdot \sum_{k=0}^n \binom{k}{1} \\ &+ 6 \cdot \sum_{k=0}^n \binom{k}{2} = 4(n+1) + 1 \cdot \binom{n+1}{2} + 6 \cdot \binom{n+1}{3} = 4(n+1) + \frac{(n+1)n}{2} + \frac{(n+1)(n)(n-1)}{3!} \cdot 6 \\ &= (n+1) \left(\frac{2n^2}{2} - n + 8 \right) / 2. \end{aligned}$$

#5(a) Function form of PHP: If $f: P \rightarrow H$ is any function & $|P| > |H|$, then there exists $a_1, a_2 \in P$ with $a_1 \neq a_2$ such that $f(a_1) = f(a_2)$.

(b) Let b be any person in P and put $A = \text{set of acquaintances of } b \text{ in } P$, and $S = \text{set of strangers to } b \text{ in } P$. Then $A \cap S = \emptyset$ and $|A \cup S| = 9$. So $|A| \geq 4$ or $|S| \geq 6$. (If $|A| \leq 3$ & $|S| \leq 5$, then we would get $|A \cup S| \leq 8$ - a contradiction).

Case(i) $|A| \geq 4$: In this case either A contains two acquaintances or everyone in A are mutual strangers. In the first case, if we add b to the 2 acq., we get 3 mut. acq. - and in the second case we easily get 4 mut. strangers, because $|A| \geq 4$.

Case(ii) $|S| \geq 6$: In this case, S is guaranteed to have 3 mut. acq. or 3 mutual strangers. In the first case, we got our 3 mut. acq. and in the 2nd case, we add b to the 3 mut. strangers to get 4 mut. strangers.

#6.(a) The Stirling numbers of the First kind are the unique integers $[n]_p$ such that $[n]_p = \sum_{k=0}^p (-1)^{p-k} [p]_k \cdot n^k$. The Stirling numbers

of the Second kind are the unique integers $\{n\}_k$ such that $n^p = \sum_{k=0}^p \{n\}_k \cdot [n]_k$. (Here $[n]_k = n(n-1)(n-2)\dots(n-(k-1))$).

(b) We know that $n^p = \sum_{k=0}^p \{n\}_k \cdot [n]_k \dots (*)$. So

$$n^p = n^{p-1} \cdot n = \left(\sum_{k=0}^{p-1} \{p-1\}_k \cdot [n]_k \right) \cdot n = \sum_{k=0}^{p-1} \{p-1\}_k \cdot [n]_k \cdot [(n-k)+k]$$

$$= \sum_{k=0}^{p-1} \{p-1\}_k \cdot [n]_{k+1} + \sum_{k=0}^{p-1} \{p-1\}_k \cdot [n]_k \cdot k$$

$$= \sum_{k=1}^p \{p-1\}_k \cdot [n]_k + \sum_{k=0}^{p-1} k \cdot \{p-1\}_k \cdot [n]_k \quad \text{by replacing } k \text{ by } k-1 \text{ in the first sum}$$

$$= \{p-1\}_p \cdot [n]_p + \underbrace{\sum_{k=1}^{p-1} (\{p-1\}_{k-1} + k \cdot \{p-1\}_k) \cdot [n]_k}_{0 \leq k < p} + 0 \cdot \{p-1\}_0 \cdot [n]_0$$

Comparing the coefficients of $[n]_k$ of this equation with those of (*) we can see that for all $1 \leq k \leq p-1$, $\{n\}_k = \{p-1\}_{k-1} + k \cdot \{p-1\}_k$. END
same as 0 $\leq k < p$