

TEST #1 - SPRING 2006TIME: 75 min.

Answer all 6 questions. Show all working and provide all reasoning. An unjustified answer will receive little credit.  
 BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.

- (15) 1. (a) How many integers in the set  $\{1, 2, 3, \dots, 2000\}$  are divisible by 15 or 20?  
 (b) Find the permutation which has  $<3, 4, 1, 2, 1, 0>$  as its inversion sequence.
- (15) 2. (a) Write down what the *Inclusion-Exclusion Theorem* says. Give all the details.  
 (b) A shop has huge numbers of Pepsi, Coke, 7-up, and Sprite sodas only. In how many ways can a lady buy a bag of 12 sodas if she wants at least one of each kind?
- (15) 3. (a) A boy lives 5 blocks west and 7 blocks south of his school. In how many ways can he walk the 12 blocks to school if his first two blocks are always eastwards (so that he can pass by his favorite tree)?  
 (b) Find the number of integer solutions of the equation  $x_1 + x_2 + x_3 = 9$  with  $x_1 \geq 3$ ,  $x_2 \geq -4$  and  $x_3 \geq 2$ .
- (20) 4. (a) Write down what the *Multinomial Theorem* says. Then find the coefficient of  $x^2y^3z$  in the expansion of  $(5x-2y+3z)^6$ .  
 (b) Use the *Binomial Theorem* to find the value of the sum
- $$\frac{1}{1} \cdot \binom{n}{0} - \frac{1}{2} \cdot \binom{n}{1} + \frac{1}{3} \cdot \binom{n}{2} - \dots + \frac{(-1)^n}{n+1} \cdot \binom{n}{n}$$
- (20) 5. (a) How many permutations of  $\{1, 2, 3, \dots, 7\}$  have exactly 3 elements going to themselves?  
 (b) How many permutations of  $\{1, 2, 3, \dots, 7\}$  have no even elements going to themselves?  
 [In part (b) you may leave your answer in terms of factorials and the binomial coefficients.]
- (15) 6. (a) Write down the general form of the *Pigeon-Hole Principle*  
 (b) Prove that in any group of 10 people we can always find 3 mutual acquaintances or 4 mutual strangers.  
 [You may use the fact that in any group of 6 people we can find 3 mutual acquaintances or 3 mutual strangers.]

1. (a) Let  $U = \{1, 2, 3, \dots, 2000\}$ ,  $A = \{x \in U : x \text{ is divisible by } 15\}$  and  $B = \{x \in U : x \text{ is divisible by } 20\}$ . Then  
 $A \cap B = \{x \in U : x \text{ is divisible by both } 15 \text{ and } 20\}$   
 $= \{x \in U : x \text{ is divisible by l.c.m. } (15, 20)\}$   
 $= \{x \in U : x \text{ is divisible by } 60\}$

So  $|A| = \left\lfloor \frac{2000}{15} \right\rfloor = \left\lfloor 133\frac{1}{3} \right\rfloor = 133$ ,  $|B| = \left\lfloor \frac{2000}{20} \right\rfloor = \left\lfloor 100 \right\rfloor = 100$ ,  
and  $|A \cap B| = \left\lfloor \frac{2000}{60} \right\rfloor = \left\lfloor 33\frac{1}{3} \right\rfloor = 33$ . Hence

No. of integers in  $U$  which are divisible by 15 or 20  
 $= |A \cup B| = |A| + |B| - |A \cap B| = 133 + 100 - 33 = 200$ .

(b) We first start by writing down 6 because the length of the inversion sequence is 6. Then we get :

6, 5 b.c. we need  $i_5 = 1$  bigger integer in front of 5

6, 5, 4 b.c. we need  $i_4 = 2$  bigger integers in front of 4

6, 3, 5, 4 b.c. we need  $i_3 = 1$  bigger integer in front of 3

6, 3, 5, 4, 2 b.c. we need  $i_2 = 4$  bigger integers in front of 2

6, 3, 5, 1, 4, 2 b.c. we need  $i_1 = 3$  bigger integers in front of 1.

So  $\langle 6, 3, 5, 1, 4, 2 \rangle$  is the required permutation.

2(a) Inclusion-Exclusion Theorem: Let  $A_1, \dots, A_n$  be subsets of the universal set  $U$ . Then

$$|A_1^c \cap A_2^c \cap \dots \cap A_n^c| = \sum_{k=0}^n (-1)^k \cdot \left\{ \sum_{\substack{(i_1, \dots, i_k) \text{ is a} \\ \text{subsequence of } \langle 1, 2, 3, \dots, n \rangle \\ \text{of length } k}} |U^{(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap \dots \cap A_{i_k})}| \right\}$$

$\langle i_1, \dots, i_k \rangle$  is a  
subsequence of  
 $\langle 1, 2, 3, \dots, n \rangle$ .  
of length  $k$ .

Note that when  $k=0$ ,  $\langle i_1, \dots, i_k \rangle = \langle \rangle =$  the empty sequence.

- : 2(b) Number of ways the lady can buy 12 sodas with at least one of each of the four kinds
- = no. of 12-comb. of  $\{\infty, a, \infty, b, \infty, c, \infty, d\}$   
 with at least one  $a, b, c$  and  $d$  each
- = no. of integer solutions of  $x_1 + x_2 + x_3 + x_4 = 12$   
 with  $x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$  and  $x_4 \geq 1$
- = no. of integer solutions of  $y_1 + y_2 + y_3 + y_4 = 8$   
 with  $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$  and  $y_4 \geq 0$
- =  $\binom{8+4-1}{4-1} = \binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} = (55)(3) = 165.$

- 3(a) Number of ways the boy can walk the 12 blocks to school
- = no. of permutations of  $[5.E, 7.N]$  which begin with EE
- = no. of permutations of  $[3.E, 7.N]$
- =  $\frac{(3+7)!}{3!7!} = \frac{10!}{3!7!} = \frac{10 \cdot 9 \cdot 8^4}{3 \cdot 2 \cdot 1} = 120.$

- (b) No. of integer solutions of " $x_1 + x_2 + x_3 = 9$ "  
 with  $x_1 \geq 3, x_2 \geq -4$ , and  $x_3 \geq 2$
- = No. of non-neg. integer solutions of  
 $"(y_1+3) + (y_2-4) + (y_3+2) = 9"$
- = No. of non-neg. integer solutions of  
 $y_1 + y_2 + y_3 = 9 - (3+2-4) = 8$
- =  $\binom{8+3-1}{3-1} = \binom{10}{2} = \frac{10 \cdot 9}{2 \cdot 1} = 45.$

- 4(a) (i) Multinomial Theorem : For any non-negative integers  $n$  and  $k$  we have
- $$(x_1 + x_2 + \dots + x_k)^n = \sum_{\langle n_1, \dots, n_k \rangle: n_1 + \dots + n_k = n} \binom{n}{n_1, \dots, n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$$

$$\begin{aligned}
 & : 4(a)(ii) \text{ Coefficient of } x^2y^3z \text{ in the expansion of } (5x-2y+3z)^6 \\
 & = \text{ coefficient of the term } \binom{6}{2,3,1} \cdot (5x)^2 \cdot (-2y)^3 \cdot (3z) \\
 & = \frac{6!}{2!3!1!} \cdot (5)^2 \cdot (-2)^3 \cdot (3) = \frac{6 \cdot 5 \cdot 4^2}{2 \cdot 1} \cdot 25 \cdot (-8) \cdot 3 \\
 & = 60 \cdot (-200) \cdot 3 = 60(-600) = -36,000.
 \end{aligned}$$

(b) From the Binomial Theorem we know that

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n = (1+x)^n$$

Integrating both sides from  $-1$  to  $0$  we get

$$\begin{aligned}
 \int_{-1}^0 \left\{ \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n \right\} dx &= \int_{-1}^0 (1+x)^n dx \\
 \therefore \left[ \frac{x}{1} \binom{n}{0} + \frac{x^2}{2} \binom{n}{1} + \frac{x^3}{3} \binom{n}{2} + \dots + \frac{x^{n+1}}{n+1} \binom{n}{n} \right]_{-1}^0 &= \left[ \frac{(1+x)^{n+1}}{n+1} \right]_{-1}^0 \\
 \therefore - \left[ \frac{(-1)}{1} \binom{n}{0} + \frac{(-1)^2}{2} \binom{n}{1} + \frac{(-1)^3}{3} \binom{n}{2} + \dots + \frac{(-1)^{n+1}}{n+1} \binom{n}{n} \right] &= \frac{1}{n+1} - \frac{0}{n+1} \\
 \therefore \frac{1}{1} \binom{n}{0} - \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} - \dots + \frac{(-1)^n}{n+1} \binom{n}{n} &= \frac{1}{n+1}.
 \end{aligned}$$

5(a) No. of perm. of  $\{1, 2, \dots, 7\}$  which have exactly 3 elements going to themselves

$$\begin{aligned}
 & = (\text{No. of ways of sending}) \cdot (\text{No. of ways of deranging}) \\
 & \quad (\text{3 elements to themselves}) \quad (\text{the other 4 elements}) \\
 & = \binom{7}{3} \cdot D_4 = \frac{7!}{3!4!} \cdot 4! \left[ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right] \\
 & = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} \cdot (24 - 24 + 12 - 4 + 1) = 7(5)(9) = 315
 \end{aligned}$$

(b) Let  $U$  = set of all permutations of  $\{1, 2, 3, \dots, 7\}$ ,

$A_2$  = set of all perms in  $U$  with 2 going to itself

$A_4$  = set " in  $U$  with 4 going to itself

and  $A_6$  = set " in  $U$  with 6 going to itself.

5(b) Then  $|U| = 7!$ ,  $|A_2| = |A_4| = |A_6| = 6!$ ,  
 $|A_2 \cap A_4| = |A_2 \cap A_6| = |A_4 \cap A_6| = 5!$  and  
 $|A_2 \cap A_4 \cap A_6| = 4!$  Hence the no. of perm. in  $U$  in  
which no even integer go to themselves  $= |A_2^c \cap A_4^c \cap A_6^c|$   
 $= |U| - |A_2| - |A_4| - |A_6| + |A_2 A_4| + |A_2 A_6| + |A_4 A_6| - |A_2 A_4 A_6|$   
 $= 7! - 3(6!) + 3(5!) - 4! = (7.6.5 - 3.6.5 + 3.5 - 1) 4!$   
 $= (4.6.5 + 14) 4! = (134) 24 = (268) 12 = 3,216.$

6(a) General Form of the P.H.P. : If  $k$  pigeons are placed in  $n$  holes, then there is a hole which contains at least  $\lfloor \frac{k-1}{n} \rfloor + 1$  pigeons.

(b) Choose any person,  $p_1$ , say, from the 10 people. Let  
 $F = \text{set of all acquaintances of } p_1$  and  
 $S = \text{set of all strangers to } p_1$ . Then  $|S \cup F| = 9$   
and  $S \cap F = \emptyset$ . So either  $|S| \geq 6$  or  $|F| \geq 4$ .

Case(i) :  $|S| \geq 6$ . In case(i) we know from previous results  
in class that there are either 3 mut. strangers or 3 mut.  
acquaintances. In the first case we add  $p_1$  to get  
4 mutual strangers. And in the second case we have  
our 3 mutual acquaintances.

Case(ii)  $|F| \geq 4$ . In case(ii) we know that there must  
be  $\binom{2}{2}$  acquaintances in  $F$  or all the people in  $F$  are  
mut. strangers. In the first case we add  $p_1$  to get  
3 mutual acquaintances. And in the second case we  
get 4 mutual strangers because  $|F| \geq 4$ .

Hence in all the cases we can always find 4 mutual  
strangers or 3 mutual acquaintances.