

TEST #2 - SPRING 2006TIME: 75 min.

Answer all 6 questions. An unjustified answer or failure to follow instructions will result in little credit. So show all working and provide all reasoning. BEGIN EACH QUESTION ON A SEPARATE PAGE.

- (15) 1. Find the solution of the equation $a_{n+2} - a_{n+1} - 6a_n = 0$ with the initial conditions $a_0=7, a_1=-4$.
- (20) 2. Find the general solution of the following difference equations
- $a_{n+2} - 6a_{n+1} + 9a_n = 8$.
 - $a_{n+2} - 4a_n = 9n$.
- (15) 3. Use the method of generating functions to find the solution of the difference equation $a_n + 2a_{n-1} + 9 = 0$ with the initial condition $a_0=2$.
- (15) 4. (a) Starting with $(1-x)^{-1} = 1 + x + x^2 + \dots + x^n + \dots$, find the generating function for $\langle 1/\{(n+1).2^n\} \rangle_{n \geq 0}$.
- (b) Let $M = [3.a, 2.b, 3.c]$. Use the method of exponential generating functions to find the number of 5-permutations of M . (Express your answers in terms of factorials and simplify as far as possible)
- (15) 5. (a) Define what is the zero diagonal (=zero column) of a sequence $\langle h_n \rangle_{n \geq 0}$.
- (b) Let $h_n = 6n^2 - 2n + 5$. So $\langle h_n \rangle = \langle 5, 9, 25, 53, 93, 145, \dots \rangle$. Find the zero diagonal of $\langle h_n \rangle$ and a formula for the sum $h_0 + h_1 + h_2 + \dots + h_n$. (Simplify your answer)
- (20) 6. (a) Define the Stirling numbers of the First kind and the Stirling numbers of the Second kind.
- (b) Starting from the definition, prove that for any k with $0 < k < p$, we have

$$\begin{bmatrix} p \\ k \end{bmatrix} = \begin{bmatrix} p-1 \\ k-1 \end{bmatrix} + (p-1) \cdot \begin{bmatrix} p-1 \\ k \end{bmatrix}.$$

$$1. \quad a_{n+2} - a_{n+1} - 6a_n = 0 \quad a_0 = 7, a_1 = -4$$

$$\therefore (E^2 - E - 6)a_n = 0 \Rightarrow (E+2)(E-3) = 0$$

$$\therefore E = -2 \text{ or } 3. \quad \text{Hence } a_n = C_1 \cdot (-2)^n + C_2 \cdot 3^n$$

But $a_0 = 7$ and $a_1 = -4$. So

$$7 = C_1 \cdot 1 + C_2 \cdot 1$$

$$-4 = C_1 \cdot (-2) + C_2 \cdot 3$$

$$\therefore C_2 = 7 - C_1, \quad \text{so} \quad -4 = C_1 \cdot (-2) + (7 - C_1) \cdot 3$$

$$\text{Hence} \quad -25 = -5C_1 \Rightarrow C_1 = 5$$

$$\therefore C_2 = 7 - 5 = 2. \quad \text{So} \quad a_n = 5 \cdot (-2)^n + 2 \cdot (3)^n$$

$$2. \quad (a) \quad a_{n+2} - 6a_{n+1} + 9a_n = 8 \quad (*)$$

$$\text{Homog. Eq is } (E^2 - 6E + 9)a_n = 0$$

$$\therefore (E-3)^2 a_n = 0 \Rightarrow a_n^c = (C_1 + C_2 \cdot n) \cdot (3)^n$$

$$\text{Try } a_n^p = B. \quad \text{Then } a_{n+1}^p = B \text{ & } a_{n+2}^p = B$$

$$\text{So } (*) \text{ becomes } B - 6B + 9B = 8$$

$$\therefore +4B = 8 \Rightarrow B = +2. \quad \therefore a_n^p = +2$$

$$\text{So general solution is } a_n = (C_1 + C_2 \cdot n) \cdot (3)^n + 2.$$

$$(b) \quad a_{n+2} - 4a_{n+1} = 9n \quad (**)$$

$$\text{Homog. Eq is } (E^2 - 4)a_n = 0$$

$$\therefore (E-2)(E+2)a_n = 0 \Rightarrow a_n^c = C_1 \cdot (2)^n + C_2 \cdot (-2)^n$$

$$\text{Try } a_n^p = An + B. \quad \text{Then } a_{n+2}^p = A(n+2) + B. \quad \text{So } (**) \text{ becomes}$$

$$A(n+2) + B - 4(An + B) = 9n$$

$$\therefore -3An + 2A - 3B = 9n$$

$$\therefore A = -3 \quad \& \quad 3B = 2A = -6 \Rightarrow B = -2$$

$$\therefore a_n = a_n^c + a_n^p = C_1 \cdot (2)^n + C_2 \cdot (-2)^n - 3n - 2.$$

$$3. \quad a_n + 2a_{n-1} + 9 = 0 \quad \& \quad a_0 = 2$$

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

Then $2x f(x) = 2a_0 x + 2a_1 x^2 + \dots + 2a_{n-1} x^n + \dots$

$$\frac{9}{1-x} = 9 + 9x + 9x^2 + \dots + 9x^n + \dots$$

$$\begin{aligned} \therefore (1+2x)f(x) + \frac{9}{1-x} &= (a_0 + 9) + (a_1 + 2a_0 + 9)x + \dots + (a_n + 2a_{n-1} + 9)x^n + \dots \\ &= (2+9) + \sum_{n=1}^{\infty} (a_n + 2a_{n-1} + 9)x^n \\ &= 11 + 0 = 11 \end{aligned}$$

$$\therefore (1+2x)f(x) = 11 + \frac{9}{1-x} = \frac{-11x+2}{1-x}$$

$$\therefore f(x) = \frac{-11x+2}{(1+2x)(1-x)} = \frac{A}{1+2x} + \frac{B}{1-x}$$

$$\therefore A(1-x) + B(1+2x) = -11x+2$$

Putting $x=1$, gives us $B(1+2) = -11 \cdot 1 + 2 \Rightarrow B = -3$

Putting $x = -\frac{1}{2}$ gives us $A(1+\frac{1}{2}) = -11(\frac{1}{2}) + 2 \Rightarrow A = 5$

$$\therefore f(x) = \frac{5}{1+2x} - \frac{3}{1-x} = \sum_{n=0}^{\infty} 5 \cdot (-2)^n x^n - \sum_{n=0}^{\infty} 3 \cdot (1)^n x^n$$

$$\therefore a_n = \text{coeff. of } x^n \text{ in the exp. of } f(x) = 5 \cdot (-2)^n - 3$$

4(a)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \therefore \quad \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

$$\therefore \int_0^{\frac{x}{2}} \left(\sum_{n=0}^{\infty} t^n \right) dt = \int_0^{\frac{x}{2}} \frac{1}{1-t} dt. \quad \sum_{n=0}^{\infty} \left[\frac{t^{n+1}}{n+1} \right]_0^{\frac{x}{2}} = \left[-\ln(1-t) \right]_0^{\frac{x}{2}}$$

$$\therefore \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^{n+1}}{n+1} = -\ln(1-\frac{x}{2}) + \ln(1)$$

$$\therefore \frac{x}{2} \cdot \sum_{n=0}^{\infty} \frac{x^n}{(n+1) \cdot 2^n} = -\ln(-\frac{x}{2}) \quad \text{because } \ln(1) = 0$$

$$\therefore \sum_{n=0}^{\infty} \frac{x^n}{(n+1) \cdot 2^n} = -\frac{2}{x} \ln\left(\frac{1-x}{2}\right)$$

So the generating function of $\frac{1}{(n+1) \cdot 2^n}$ is $-\frac{2}{x} \ln\left(\frac{1-x}{2}\right)$.

4(b) $M = [3.a, 2.b, 3.c]$. The number of 5-permutations of M is the coefficient of $\frac{x^5}{5!}$ in the exponential expansion of

$$\underbrace{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)}_{\# \text{ of } a's} \underbrace{\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)}_{\# \text{ of } b's} \underbrace{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)}_{\# \text{ of } c's}.$$

$$1. \frac{x^2}{1!} \cdot \frac{x^3}{3!} + \frac{x}{1!} \cdot \frac{x}{1!} \cdot \frac{x^3}{3!} + \frac{x}{1!} \cdot \frac{x^2}{2!} \cdot \frac{x^2}{2!} + \frac{x^2}{2!} \cdot \frac{x}{1!} \cdot \frac{x^3}{3!} \\ + \frac{x^2}{2!} \cdot \frac{x}{1!} \cdot \frac{x^2}{2!} + \frac{x^2}{2!} \cdot \frac{x^2}{2!} \cdot \frac{x}{1!} + \frac{x^3}{3!} \cdot \frac{1}{1!} \cdot \frac{x^2}{2!} + \frac{x^3}{3!} \cdot \frac{x}{1!} \cdot \frac{x}{1!} + \frac{x^3}{3!} \cdot \frac{x^2}{2!} \cdot 1$$

$$\text{Ans:} = \left(\frac{2}{1!} \cdot \frac{1}{1!} \cdot \frac{1}{3!} + \frac{4}{2!} \cdot \frac{1}{1!} \cdot \frac{1}{3!} + \frac{3}{1!} \cdot \frac{1}{2!} \cdot \frac{1}{2!} \right) \cdot 5! \\ = \left[\frac{2}{6} + \frac{4}{2(6)} + \frac{3}{(2)(2)} \right] (120) = \frac{17}{12} \cdot 120 = 170$$

5(a) The zero diagonal of the sequence $\langle h_n \rangle_{n \geq 0}$ is the sequence $\langle \Delta^n h_0 \rangle_{n \geq 0}$ where $\Delta h_n = h_{n+1} - h_n$ and $\Delta^{k+1} h_n = \Delta(\Delta^k h_n)$ for $k \geq 0$.

(b) $h_n = 6n^2 - 2n + 5$

n	0	1	2	3	4	5
h_n	5	9	25	53	93	145
Δh_n	4	16	28	40	52	
$\Delta^2 h_n$	12	12	12	12		
$\Delta^3 h_n$	0	0	0			
$\Delta^4 h_n$	0	0				

So zero diagonal = $\langle 5, 4, 12, 0, 0, \dots \rangle$

Hence $h_k = 5 \cdot \binom{k}{0} + 4 \cdot \binom{k}{1} + 12 \cdot \binom{k}{2}$

$$\therefore \sum_{k=0}^n h_k = 5 \cdot \binom{n+1}{1} + 4 \cdot \binom{n+1}{2} + 12 \cdot \binom{n+1}{3}$$

$$= 5 \cdot \frac{(n+1)!}{n! 1!} + 4 \cdot \frac{(n+1)!}{(n-1)! 2!} + \frac{12 \cdot (n+1)!}{3! (n-2)!}$$

$$= 5 \cdot (n+1) + \frac{4}{2} (n+1)n + \frac{12}{6} \cdot (n+1)n(n-1) = (n+1)(n^2 + 5n + 5)$$

#6 (a) The Stirling numbers of the first kind are the unique integers $[p]_k$ such that $[n]_p = \sum_{k=0}^p (-1)^{p-k} [p]_k \cdot n^k$.

The Stirling numbers of the second kind are the unique integers $\{p\}_k$ such that $n^p = \sum_{k=0}^p \{p\}_k \cdot [n]_k$.

These numbers are defined for all $p \geq 0, k \geq 0$,

(b) From the definition we know that

$$(*) \quad [n]_p = \sum_{k=0}^p (-1)^{p-k} \cdot \begin{Bmatrix} p \\ k \end{Bmatrix} \cdot n^k, \quad \text{so}$$

$$[n]_{p-1} = \sum_{k=0}^{p-1} (-1)^{p-1-k} \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix} \cdot n^k.$$

$$\begin{aligned} \therefore [n]_p &= [n]_{p-1} \cdot [n-(p-1)] = [n-(p-1)] \cdot [n]_{p-1} \\ &= [n-(p-1)] \cdot \sum_{k=0}^{p-1} (-1)^{p-1-k} \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix} \cdot n^k \\ &= \sum_{k=0}^{p-1} (-1)^{p-1-k} \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix} \cdot n^{k+1} - (p-1) \sum_{k=0}^{p-1} (-1)^{p-1-k} \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix} n^k \\ &= \underbrace{\sum_{k=1}^p (-1)^{p-k} \cdot \begin{Bmatrix} p-1 \\ k-1 \end{Bmatrix} \cdot n^k}_{\text{replace } k \text{ by } k-1} + \underbrace{\sum_{k=0}^{p-1} (-1)^{p-k} \cdot (p-1) \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix} n^k}_{\text{The " - " went into here}} \\ &= (-1)^{p-p} \cdot \begin{Bmatrix} p-1 \\ p-1 \end{Bmatrix} n^p + \sum_{k=1}^{p-1} (-1)^{p-k} \cdot \left\{ \begin{Bmatrix} p-1 \\ k-1 \end{Bmatrix} + (p-1) \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix} \right\} n^k \\ &\quad + (-1)^{p-0} \cdot (p-1) \cdot \begin{Bmatrix} p-1 \\ 0 \end{Bmatrix} n^0 \end{aligned}$$

Comparing the coefficients of n^k for $0 \leq k < p$ in this equation with that of (*) we see that

$$\begin{Bmatrix} p \\ k \end{Bmatrix} = \begin{Bmatrix} p-1 \\ k-1 \end{Bmatrix} + (p-1) \cdot \begin{Bmatrix} p-1 \\ k \end{Bmatrix} \quad \text{for } 0 \leq k < p.$$

Note that we also get from taking $k=0$ & $k=p$ that

$$\begin{Bmatrix} p \\ p \end{Bmatrix} = \begin{Bmatrix} p-1 \\ p-1 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} p \\ 0 \end{Bmatrix} = (p-1) \cdot \begin{Bmatrix} p-1 \\ 0 \end{Bmatrix}$$

For all of this to make sense we must have that $p \geq 1$.